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Twists in $U(\mathfrak{sl}_3)$ and their quantizations

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Abstract
The solutions of the Drinfeld equation corresponding to the full set of different carrier subalgebras $\mathfrak{f} \subset \mathfrak{sl}_3$ are explicitly constructed. The obtained Hopf structures are studied. It is demonstrated that the presented twist deformations can be considered as limits of the corresponding quantum analogues (q-twists) defined for the q-quantized algebras.

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1. Introduction

Triangular Hopf algebras $\mathcal{A}(m, \Delta, S, \eta, \epsilon; \mathcal{R})$ [1] play an essential role in quantum theory and in particular for models with noncommutative spacetime [2, 3]. Quantizations of antisymmetric r-matrices (solutions of the classical Yang–Baxter equation) form an important class of such algebras. They describe Poisson structures compatible with the initial Lie algebra $\mathfrak{g}$, i.e. the mechanical systems that can exist on a space whose noncommutativity is fixed by $\mathfrak{g}$. Such quantum algebras can be constructed in terms of r-matrices by means of Campbell–Hausdorff series [4]. However, these constructions are obviously inappropriate for an efficient usage of quantum $\mathcal{R}$-matrices. If one provides the elements of the initial Lie algebra $\mathfrak{g}$ with primitive coproducts Δ^{prim} and consider the universal enveloping algebra $U(\mathfrak{g})$ as a Hopf algebra with the costructure generated by Δ^{prim}, then the solution $\mathcal{F} \in U(\mathfrak{g})^{\otimes 2}$ of the twist equation [4]

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{F}), \tag{1}$$

$$(\epsilon \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \epsilon)\mathcal{F} = 1 \tag{2}$$

allows one to find the solution of the Yang–Baxter equation, namely $\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{F}^{-1}$. Thus, to obtain the set of solutions of the twist equations (1) (the set of twists $\mathcal{F}$) for the full set of carrier subalgebras in simple Lie algebras $\mathfrak{g}$ used in constructing physical models is an important task (see for example [5] and references therein). Such a set of twists for a Lie algebra $\mathfrak{g}$ will be considered complete if for any class $\mathfrak{h}$ of equivalent antisymmetric solutions for the classical Yang–Baxter equation (CYBE) we can attribute a twist $\mathcal{F}(\xi)$ (with a deformation parameter ξ) whose classical r-matrix,

$$r_{\mathcal{F}} = \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}(\xi)_{21}\mathcal{F}(\xi)^{-1}|_{\xi=0},$$

is a representative of the class $\mathfrak{h}$.

Each r-matrix induces a dual map that can be treated as a skew-symmetric bilinear form ω_r that satisfies the condition

$$\omega(x,[y,z]) + \omega(y,[z,x]) + \omega(z,[x,y]) = 0.$$

Form ω_r is nondegenerate on the space of a subalgebra $\mathfrak{g}_c \subseteq \mathfrak{g}$. Such a subalgebra $\mathfrak{g}_c$ is called the carrier of r. Subalgebras supplied with nondegenerate form ω_r are called Lie quasi-Frobenius. The classification problem for quasi-Frobenius Lie subalgebras is far from being completed. The explicit classification is known for some types of Lie algebras and in particular for $\mathfrak{sl}_3$ it was given by Stolin [6].

The construction of the twisting elements is not only important but also a difficult problem and for a long time only a few types of twists were known in an explicit form [7–10]. In this paper, we demonstrate that using the factorization property [11, 12] of twists the explicit solution of equations (1) and (2) can be constructed for any quasi-Frobenius subalgebra in $\mathfrak{sl}_3$. In section 2, the corresponding Hopf algebras—twist deformations $U(\mathfrak{sl}_3) \longrightarrow U_F(\mathfrak{sl}_3)$—are classified and studied.

The second fundamental problem in the study of triangular Hopf algebras is the relation between twist deformations and ordinary quantizations (q-deformations). In section 3, we study the possibility of attributing to each twist $\mathcal{F}$ (and the corresponding deformation $U_{\mathcal{F}}(\mathfrak{f})$) the quantum twist F'_q defining deformation of quantized current algebra $U_q(\mathfrak{sl}_3)$ and specializing to a twist $\mathcal{F}$ of $U(\mathfrak{sl}_3)$ in the limit $q \to 1$. The limit is assumed to be taken along some curve $q = 1 + f(s)$ as $f(s) \to 0$ for $s \to 0$. In particular, we can obtain quantum versions of the quasi-Frobenius subalgebras through commutativity of the following diagram:

$$\begin{array}{ccccc} U(\mathfrak{f}) & \xrightarrow{q} & U_q(\mathfrak{f}) & \xrightarrow{\iota_q} & U_q(\widehat{\mathfrak{sl}}_3) \\ \mathcal{F}\downarrow & & F_q\downarrow & & F'_q\downarrow \\ U_{\mathcal{F}}(\mathfrak{f}) & \xrightarrow{q} & U_{F_q}(\mathfrak{f}) & \xrightarrow{\iota_q} & U_{F'_q}(\widehat{\mathfrak{sl}}_3), \end{array} \tag{3}$$

where ι_q is the embedding, $\widehat{\mathfrak{sl}}_3$ stands for the affinization of $\mathfrak{sl}_3$: $A_2^{(1)}$ or $A_2^{(2)}$ and $F'_q := (\iota_q \otimes \iota_q)F_q\left(\iota_q^{-1} \otimes \iota_q^{-1}\right)$. Formally, our quantization $U_q(\mathfrak{sl}_3)$ is defined over $\mathbb{C}[s, s^{-1}]$ and the specialization $q \to 1$ corresponds to the limit $s \to 0$. In section 3, we step by step demonstrate that in most cases one can straightforwardly obtain $U_q(\mathfrak{f})$ as a Hopf subalgebra of $U_q^{\mathcal{J}}(\widehat{\mathfrak{sl}}_3)$, the Drinfeld–Jimbo quantization of $U(\widehat{\mathfrak{sl}}_3)$ with the comultiplication deformed by some factor $\mathcal{J}$:

$$\Delta_{\mathcal{J}}(x) = \mathcal{J}\Delta_q(x)\mathcal{J}^{-1}.$$

$U_q(\mathfrak{f})$ is fixed by the fact that it is a minimal Hopf subalgebra in $U_q(\mathfrak{sl}_3)$ with the property

$$U_q(\mathfrak{f})|_{\mathbb{C}[s]} \to U(\mathfrak{f}), \qquad s \to 0.$$

In the particular case of $\mathfrak{b}^{(0)} = \{H_{13}, E_{12}+E_{23}\}$, the Borel subalgebra belongs to the subalgebra $\mathfrak{sl}_2$ that is the special subalgebra of $\mathfrak{sl}_3$ and we can define the corresponding q-quantization of $U(\mathfrak{b}^{(0)})$ by embedding it into $U_q\left(A_2^{(2)}\right)$. To define the q-twists corresponding to the twists described in section 2, we utilize an assumption that most of q-twists can be built out of

q-exponentials and Abelian twists. The only twist deformation of $U(\mathfrak{sl}_3)$ that seems to contradict this assumption is that corresponding to the r-matrix $r_{\mathcal{JR}} = \frac{1}{2}H_{23}\wedge E_{23}+\eta E_{12}\wedge E_{13}$ (see formulae (30)). In the appendix, the special properties of the so-called peripheric twists [13] are discussed.

2. Classification of quasi-Frobenius subalgebras in $\mathfrak{sl}_3$ and twist deformations $U_F(\mathfrak{sl}_3)$

2.1. Abelian two-dimensional subalgebras

We have four classes of nonequivalent two-dimensional subalgebras [6] denoted by

$$\mathfrak{h} \quad \Longrightarrow \quad H = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix},$$

$$\mathfrak{h}^{(1)} \quad \Longrightarrow \quad X = * \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{h}^{(0,1)} \quad \Longrightarrow \quad X = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{h}^{(1,1)} \quad \Longrightarrow \quad X = * \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2.2. Cartan subalgebra: case $\mathfrak{h}$

Any two elements $H_1, H_2 \in \mathbf{H}$ form a carrier subalgebra for the so-called Abelian twist (first described by Reshetikhin [7]):

$$\mathcal{F}_{\mathcal{R}} = \exp(\xi^{12} H_1 \otimes H_2), \qquad \xi^{12} \in \mathbb{C}. \tag{4}$$

Let Λ be the root system of $\mathfrak{sl}_3$. For any $\lambda \in \Lambda$, let $E_\lambda \in \mathfrak{sl}_3$ be the element corresponding to the root λ. The twisting elements $\mathcal{F}_{\mathcal{R}}$ lead to the deformed costructure

$$\begin{aligned} \Delta_{\mathcal{F}_{\mathcal{R}}}(H_{1,2}) &= H_{1,2} \otimes 1 + 1 \otimes H_{1,2}, \\ \Delta_{\mathcal{F}_{\mathcal{R}}}(E_\lambda) &= E_\lambda \otimes \mathrm{e}^{\xi^{12}\lambda(H_1)H_2} + \mathrm{e}^{\xi^{12}\lambda(H_2)H_1} \otimes E_\lambda. \end{aligned} \tag{5}$$

2.3. Mixed: case $\mathfrak{h}^{(1)}$

For any $E_\lambda, \lambda \in \Lambda(\mathfrak{g})$, consider the Cartan element $H_\lambda^\perp$ whose dual $\left(H_\lambda^\perp\right)^*$ (with respect to the Killing form) is a vector orthogonal to λ. This pair generates the two-dimensional Abelian algebra, the carrier for the twist:

$$\mathcal{F}_{\mathcal{R}} = \exp\left(\xi H_\lambda^\perp \otimes E_\lambda\right). \tag{6}$$

Note that here the parameter ξ can be scaled by a similarity transformation.

$$\begin{aligned} \Delta_{\mathcal{F}_{\mathcal{R}}}(H) &= H \otimes 1 + 1 \otimes H - \xi\lambda(H) H_\lambda^\perp \otimes E_\lambda, \qquad H \in \mathbf{H}, \\ \Delta_{\mathcal{F}_{\mathcal{R}}}(E_\mu)|_{\mu\neq-\lambda} &= E_\mu \otimes \mathrm{e}^{\xi\mu(H_\lambda^\perp)E_\lambda} + 1 \otimes E_\mu + \xi H_\lambda^\perp \otimes E_{\lambda+\mu}, \\ \Delta_{\mathcal{F}_{\mathcal{R}}}(E_{-\lambda}) &= E_{-\lambda} \otimes 1 + 1 \otimes E_{-\lambda} + \xi H_\lambda^\perp \otimes H_\lambda - \xi^2 \left(H_\lambda^\perp\right)^2 \otimes E_\lambda. \end{aligned} \tag{7}$$

(It is assumed that $E_{\{\lambda+\mu\}}$ is zero iff $\lambda + \mu$ is not a root.)

2.4. $\mathbf{N}^+$ *subalgebra: cases* $\mathfrak{h}^{(0,1)}$ *and* $\mathfrak{h}^{(1,1)}$

Consider, for example, the commuting generators $E_{12} + \mu E_{23}$, E_{13} and the corresponding Reshetikhin twist

$$\mathcal{F}_{\mathcal{R}} = \exp(\xi (E_{12} + \mu E_{23}) \otimes E_{13}). \tag{8}$$

Here the cases $\mu = 0$ and $\mu \neq 0$ are to be considered separately. The twist $\mathcal{F}_{\mathcal{R}}(\xi, \mu = 0)$ cannot be scaled to $\mathcal{F}_{\mathcal{R}}(\xi, \mu \neq 0)$. Though, it is a limit for the family $\{\mathcal{F}_{\mathcal{R}}(\xi, \mu)\}$ as is clearly seen from the explicit costructure below.

$$\Delta_{\mathcal{F}_{\mathcal{R}}}(E_{12} + \mu E_{23}) = (E_{12} + \mu E_{23}) \otimes 1 + 1 \otimes (E_{12} + \mu E_{23}), \tag{9}$$

$$\begin{aligned}
\Delta_{\mathcal{F}_{\mathcal{R}}}(E_{13}) &= E_{13} \otimes 1 + 1 \otimes E_{13},\\
\Delta_{\mathcal{F}_{\mathcal{R}}}(H_{12}) &= H_{12} \otimes 1 + 1 \otimes H_{12} - 3\xi E_{12} \otimes E_{13} + \tfrac{3}{2}\xi^2 \mu E_{13} \otimes (E_{13})^2,\\
\Delta_{\mathcal{F}_{\mathcal{R}}}(H_{23}) &= H_{23} \otimes 1 + 1 \otimes H_{23} - 3\xi \mu E_{23} \otimes E_{13} - \tfrac{3}{2}\xi^2 \mu E_{13} \otimes (E_{13})^2,\\
\Delta_{\mathcal{F}_{\mathcal{R}}}(E_{21}) &= E_{21} \otimes 1 + 1 \otimes E_{21} + \xi H_{12} \otimes E_{13} - \xi (E_{12} + \mu E_{23}) \otimes E_{23}\\
&\quad - \tfrac{1}{2}\xi^2 (2E_{12} - \mu E_{23}) \otimes (E_{13})^2 + \tfrac{1}{2}\xi^3 \mu E_{13} \otimes (E_{13})^3,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\Delta_{\mathcal{F}_{\mathcal{R}}}(E_{23}) &= E_{23} \otimes 1 + 1 \otimes E_{23} + \xi E_{13} \otimes E_{13},\\
\Delta_{\mathcal{F}_{\mathcal{R}}}(E_{32}) &= E_{32} \otimes 1 + 1 \otimes E_{32} + \xi \mu H_{23} \otimes E_{13} + \tfrac{1}{2}\xi^2 \mu (E_{12} - 2\mu E_{23}) \otimes (E_{13})^2\\
&\quad + \xi (E_{12} + \mu E_{23}) \otimes E_{12} - \tfrac{1}{2}\xi^3 \mu^2 E_{13} \otimes (E_{13})^3,\\
\Delta_{\mathcal{F}_{\mathcal{R}}}(E_{31}) &= E_{31} \otimes 1 + 1 \otimes E_{31} + \xi (-E_{32} + \mu E_{21}) \otimes E_{13} + \xi (E_{12} + \mu E_{23}) \otimes H_{13}\\
&\quad + \tfrac{1}{2}\xi^2 \mu (H_{12} - H_{23}) \otimes (E_{13})^2 - \xi^2 (E_{12} + \mu E_{23})^2 \otimes E_{13} +\\
&\quad + \tfrac{1}{2}\xi^3 \mu (-E_{12} + \mu E_{23}) \otimes (E_{13})^3 + \tfrac{1}{2}\xi^4 \mu^2 E_{13} \otimes (E_{13})^4.
\end{aligned} \tag{11}$$

2.5. $\mathbf{B}(2)$ *subalgebras and Jordanian twists*

In this section, the carrier algebra is normalized as

$$[H, E] = E, \tag{12}$$

and the twist is Jordanian [8]:

$$\mathcal{F}_{\mathcal{J}} = \exp(H \otimes \sigma(\xi)), \qquad \sigma(\xi) = \ln(1 + \xi E), \tag{13}$$

the parameter can be scaled by $\mathrm{ad}(H)$. Our task is to enumerate the inequivalent $\mathbf{B}(2)$ subalgebras.

Choose the generator E in $\mathbf{N}^+$. We must distinguish the cases where the Cartan generator for $\mathbf{B}(2)$ can be diagonalized in $\mathbf{N}^+$ and where it cannot. Only two-dimensional eigenspaces (for a Cartan subalgebra) could be found in $\mathbf{N}^+$. Note that one can always add E to H without principal influence on the results; we shall not consider such cases separately.

2.5.1. *Irregular element H and $E = E_{13}$.*

It is sufficient to specialize H as follows:

$$H = a^i E_{ii} + \alpha E_{23} + \beta E_{12}, \qquad \sum a^i = 0. \tag{14}$$

The coefficients α and β can be scaled by an appropriate $\mathrm{ad}(H^{\perp})$. We can have only the element proportional to $E = \gamma E_{13}$ here. The answer is a continuous family of $\mathbf{B}(2)$ subalgebras:

$$H = \mu E_{11} + (1 - 2\mu) E_{22} + (\mu - 1) E_{33} + \alpha E_{23} + \beta E_{12}, \qquad E = E_{13}. \tag{15}$$

Up to equivalence, the corresponding Jordanian twist is represented by the following expression:

$$\mathcal{F}_{irr} = \exp\left(\left(H_{12}^{\perp} + \eta E_{12}\right) \otimes \sigma_{13}\right).$$

The twisted coproducts are

$$\Delta(H_{12}) = H_{12} \otimes 1 - 2\eta E_{12} \otimes \sigma_{13} - \left(H_{12}^{\perp} + \eta E_{12}\right) \otimes (1 - \mathrm{e}^{-\sigma_{13}}) + 1 \otimes H_{12},$$
$$\Delta(H_{23}) = H_{23} \otimes 1 + \eta E_{12} \otimes \sigma_{13} - \left(H_{12}^{\perp} + \eta E_{12}\right) \otimes (1 - \mathrm{e}^{-\sigma_{13}}) + 1 \otimes H_{23},$$
$$\Delta(E_{12}) = E_{12} \otimes 1 + 1 \otimes E_{12},$$
$$\Delta(E_{23}) = E_{23} \otimes \mathrm{e}^{\sigma_{13}} + \eta E_{13} \otimes \sigma_{13}\mathrm{e}^{\sigma_{13}} + 1 \otimes E_{23},$$
$$\Delta(E_{13}) = E_{13} \otimes \mathrm{e}^{\sigma_{13}} + 1 \otimes E_{13},$$
$$\Delta(E_{32}) = E_{32} \otimes \mathrm{e}^{-\sigma_{13}} + \left(H_{12}^{\perp} + \eta E_{12}\right) \otimes E_{12}\,\mathrm{e}^{-\sigma_{13}} + 1 \otimes E_{32},$$
$$\Delta(E_{21}) = E_{21} \otimes 1 + \eta H_{12} \otimes \sigma_{13} - \eta^2 E_{12} \otimes (\sigma_{13})^2 - \left(H_{12}^{\perp} + \eta E_{12}\right) \otimes E_{23}\,\mathrm{e}^{-\sigma_{13}} + 1 \otimes E_{21},$$
$$\Delta(E_{31}) = E_{31} \otimes \mathrm{e}^{-\sigma_{31}} - \eta E_{32} \otimes \sigma_{13}\,\mathrm{e}^{-\sigma_{13}} + \left(H_{12}^{\perp} + \eta E_{12}\right) \otimes H_{13}\,\mathrm{e}^{-\sigma_{13}}$$
$$+ \left(\left(H_{12}^{\perp} + \eta E_{12}\right) - \left(H_{12}^{\perp} + \eta E_{12}\right)^2\right) \otimes (\mathrm{e}^{-\sigma_{13}} - \mathrm{e}^{-2\sigma_{13}}) + 1 \otimes E_{31}.$$

Thus, the irregularity results in the appearance of σ_{13} in the coproducts. When both parameters α and β are zero, we come to the regular case treated below.

2.5.2. Regular H. Let

$$H = a^i E_{ii}, \qquad \sum a^i = 0. \tag{16}$$

In the case $E = E_{13}$, all the properties of the Jordanian twist are the same as described above.

(i) $E = E_\lambda$, with λ being one of the simple roots. Let $E = E_{12}$. Again we find the parameterized family of subalgebras (and correspondingly the twists):

$$\{H = \mu E_{11} + (\mu - 1)E_{22} + (1 - 2\mu)E_{33}, E_{12}\}. \tag{17}$$

The case $E = E_{23}$ is treated analogously.

(ii) $E = E_{12} + E_{23}$. The case $E = E_{12} + \gamma E_{23}$ can be scaled to the normalized one: $E = E_{12} + E_{23}$. Immediately, we find $a^1 = 1, a^2 = 0$, thus

$$\{H = E_{11} - E_{33}, E = E_{12} + E_{23}\}. \tag{18}$$

(iii) $E = E_{12} + E_{13}$. Again only the normalized combination is to be considered. The algebra is unique:

$$\left\{H = \tfrac{2}{3}E_{11} - \tfrac{1}{3}E_{22} - \tfrac{1}{3}E_{33}, E = E_{12} + E_{13}\right\}. \tag{19}$$

The alternative combination $E = E_{23} + E_{13}$ is treated analogously.

2.6. Four-dimensional carriers

2.6.1. One Cartan generator. In this case, we can assume that the carrier belongs to the Borel subalgebra: $\mathbf{L} \subset \mathbf{B}$. Let $H_{\lambda=e_1-e_3}^{\perp} \equiv H_{13}^{\perp}$ (with the dual vector $(H_{13}^{\perp})^*$ orthogonal to the highest root $e_1 - e_3$) and assume that $a^1 \neq a^3$ in $H = a^i E_{ii}$.

In this case, the family of carrier algebras $L(\alpha, \beta, \gamma, \delta) \subset \mathbf{B}$,

$$\begin{aligned} &[H, A] = \alpha A, && [H, E] = \delta E, \\ &[H, B] = \beta B, && [A, B] = \gamma E, \\ &\alpha + \beta = \delta, && \alpha, \beta, \gamma, \delta \in \mathbb{C}, \end{aligned} \tag{20}$$

a representative can be chosen with $\delta = 1$ while γ will finally coincide with the deformation parameter. So we are to consider only the case $L(\alpha, \beta)$:

$$\begin{aligned} &[H, A] = \alpha A, \qquad [H, E] = E, \\ &[H, B] = \beta B, \qquad [A, B] = E, \\ &\alpha + \beta = 1. \end{aligned} \tag{21}$$

The set $\{\mathbf{L}_{\alpha,\beta}\}$ of carriers (21) is to be further classified due to the values of the second cohomology group $H^2(\mathbf{L}, \mathbb{C})$ and the orbits of the normalizer $N(\mathbf{L})$ (of $\mathbf{L}$ in $\mathfrak{sl}_3$) formed by its adjoint action in $H^2(\mathbf{L}, \mathbb{C})$ [6]. Consider the list of cohomological properties of $\mathbf{L}$:

	$\dim Z^2$	$\dim B^2$	$\dim H^2$
$\alpha, \beta \neq 0, -1$	3	3	0
$\left.\begin{matrix}\alpha = 0, \beta = 1 \\ \alpha = 1, \beta = 0\end{matrix}\right\}$	3	2	1
$\left.\begin{matrix}\alpha = -1, \beta = 2 \\ \alpha = 2, \beta = -1\end{matrix}\right\}$	4	3	1

(22)

The first column describes the subsets in $\{\mathbf{L}_{\alpha,\beta}\}$ that are to be considered separately.

(i) The case $\alpha, \beta \neq 0, -1$. The carrier is the Frobenius subalgebra with the nondegenerate coboundary $\omega = E^*([,])$. The corresponding twist is the extended Jordanian twist [10]; it can be written in two (equivalent) forms:

$$\begin{aligned} \mathcal{F}_{\mathcal{E}} &= \exp(\xi A \otimes B\, \mathrm{e}^{-\beta\sigma(\xi)}) \exp(H \otimes \sigma(\xi)), \\ \mathcal{F}_{\mathcal{E}'} &= \exp(-\xi B \otimes A \mathrm{e}^{-\alpha\sigma(\xi)}) \exp(H \otimes \sigma(\xi)). \end{aligned} \tag{23}$$

They are connected by the automorphism

$$i: \quad \left\{ \begin{matrix} A \longrightarrow -B \\ B \longrightarrow A \\ \alpha \rightleftarrows \beta \end{matrix} \right\}.$$

For example in the case of $\mathcal{F}_{\mathcal{E}'}$, the costructure is defined by the relations

$$\begin{aligned} \Delta_{\mathcal{E}}(H) &= H \otimes \mathrm{e}^{-\sigma(\xi)} + 1 \otimes H - \xi A \otimes B\, \mathrm{e}^{-(\beta+1)\sigma(\xi)}, \\ \Delta_{\mathcal{E}}(A) &= A \otimes \mathrm{e}^{-\beta\sigma(\xi)} + 1 \otimes A, \\ \Delta_{\mathcal{E}}(B) &= B \otimes \mathrm{e}^{\beta\sigma(\xi)} + \mathrm{e}^{\sigma(\xi)} \otimes B, \\ \Delta_{\mathcal{E}}(E) &= E \otimes \mathrm{e}^{\sigma(\xi)} + 1 \otimes E. \end{aligned} \tag{24}$$

The $\mathcal{R}$-matrix has the form

$$\begin{aligned} \mathcal{R}_{\mathcal{E}} &= \exp(\xi B\, \mathrm{e}^{-\beta\sigma(\xi)} \otimes A) \exp(\sigma(\xi) \otimes H) \exp(-H \otimes \sigma(\xi)) \exp(-\xi A \otimes B\, \mathrm{e}^{-\beta\sigma(\xi)}) \\ &= 1 \otimes 1 - \xi r_{\mathcal{E}} + \mathcal{O}(\xi^2). \end{aligned} \tag{25}$$

The corresponding classical r-matrix is

$$r_{\mathcal{E}} = H \wedge E + A \wedge B. \tag{26}$$

The parameter ξ in (23) can be scaled. Note that equal coefficients in two terms of the $r_{\mathcal{E}}$-matrix is the necessary and sufficient condition for the corresponding form $\omega_{\mathcal{E}}$ to be a cocycle. The reason is that the extension factors in $\mathcal{F}_{\mathcal{E}}$ and $\mathcal{F}_{\mathcal{E}'}$ are the discrete twists and can only borrow the continuous parameter from the smooth set of Hopf algebras (twisted by $\mathcal{F}_{\mathcal{J}} = \exp(H \otimes \sigma(\xi))$).

When the carrier $\mathbf{L}_{\alpha,\beta}$ is identified with the subalgebra of $sl(3)$, it is convenient to describe the freedom in its definition by introducing the second Cartan generator $H_{13}^{\perp} = \frac{1}{3}E_{11} - \frac{2}{3}E_{22} + \frac{1}{3}E_{33}$ and the parameter $\zeta = \alpha - \frac{1}{2}$. In these terms, the twisting element $\mathcal{F}_{\mathcal{E}}$ from (23) takes the form

$$\mathcal{F}_{\mathcal{E}} = \exp\left(\xi E_{12} \otimes E_{23}\,\mathrm{e}^{(\zeta-\frac{1}{2})\sigma(\xi)}\right)\exp\left(\left(\tfrac{1}{2}H_{13} + \zeta H_{13}^{\perp}\right)\otimes\sigma(\xi)\right).$$

The deformed $U_{\mathcal{E}}\,(sl(3)$ is defined by the following coproducts:

$$\begin{aligned}
\Delta_{\mathcal{E}}(H_{12}) &= H_{12}\otimes 1 + 1\otimes H_{12} + \left(\tfrac{1}{2}H_{13}^{\perp} + \zeta H_{13}^{\perp}\right)\otimes(\mathrm{e}^{-\sigma(\xi)} - 1)\\
&\quad - \xi E_{12}\otimes E_{23}\,\mathrm{e}^{(\zeta-\frac{3}{2})\sigma(\xi)},\\
\Delta_{\mathcal{E}}(H_{23}) &= H_{23}\otimes 1 + 1\otimes H_{23} + \left(\tfrac{1}{2}H_{13} + \zeta H_{13}^{\perp}\right)\otimes(\mathrm{e}^{-\sigma(\xi)} - 1)\\
&\quad - \xi E_{12}\otimes E_{23}\,\mathrm{e}^{(\zeta-\frac{3}{2})\sigma(\xi)},\\
\Delta_{\mathcal{E}}(E_{12}) &= E_{12}\otimes \mathrm{e}^{(\zeta-\frac{1}{2})\sigma(\xi)} + 1\otimes E_{12},\\
\Delta_{\mathcal{E}}(E_{23}) &= E_{23}\otimes \mathrm{e}^{(\frac{1}{2}-\zeta)\sigma(\xi)} + \mathrm{e}^{\sigma(\xi)}\otimes E_{23},\\
\Delta_{\mathcal{E}}(E_{13}) &= E_{13}\otimes \mathrm{e}^{\sigma(\xi)} + 1\otimes E_{13},
\end{aligned}\tag{27}$$

$$\begin{aligned}
\Delta_{\mathcal{E}}(E_{21}) &= E_{21}\otimes \mathrm{e}^{-(\frac{1}{2}+\zeta)\sigma(\xi)} + 1\otimes E_{21}\\
&\quad + \xi\left(H_{12} - \tfrac{1}{2}H_{13} - \zeta H_{13}^{\perp}\right)\otimes E_{23}\,\mathrm{e}^{-\sigma(\xi)},\\
\Delta_{\mathcal{E}}(E_{32}) &= E_{32}\otimes\ \mathrm{e}^{(\zeta-\frac{1}{2})\sigma(\xi)} + 1\otimes E_{32} + \xi E_{12}\otimes H_{23}\,\mathrm{e}^{(\zeta-\frac{1}{2})\sigma(\xi)}\\
&\quad + \xi\left(\tfrac{1}{2}H_{13} + \zeta H_{13}^{\perp}\right)\otimes E_{12}\,\mathrm{e}^{-\sigma(\xi)}\\
&\quad - \xi\left(\tfrac{1}{2}H_{13} + \zeta H_{13}^{\perp}\right)E_{12}\otimes\left(\mathrm{e}^{(\zeta-\frac{1}{2})\sigma(\xi)} - \mathrm{e}^{(\zeta-\frac{3}{2})\sigma(\xi)}\right)\\
&\quad - \xi^2 E_{12}\otimes E_{23}E_{12}\,\mathrm{e}^{(\zeta-\frac{3}{2})\sigma(\xi)} - \xi^2 E_{12}^2\otimes E_{23}\,\mathrm{e}^{2(\zeta-\frac{1}{2})\sigma(\xi)},\\
\Delta_{\mathcal{E}}(E_{31}) &= E_{31}\otimes \mathrm{e}^{-\sigma(\xi)} + 1\otimes E_{31} + \xi\left(\tfrac{1}{2}H_{13} + \zeta H_{13}^{\perp}\right)\otimes H_{13}\,\mathrm{e}^{-\sigma(\xi)}\\
&\quad + \xi\left(1 - \tfrac{1}{2}H_{13} - \zeta H_{13}^{\perp}\right)\left(\tfrac{1}{2}H_{13} + \zeta H_{13}^{\perp}\right)\otimes(\mathrm{e}^{-\sigma(\xi)} - \mathrm{e}^{-2\sigma(\xi)})\\
&\quad + \xi^2\left(\tfrac{1}{2}H_{13} + \zeta H_{13}^{\perp} - 1\right)E_{12}\otimes E_{23}\left(\mathrm{e}^{(\zeta-\frac{3}{2})\sigma(\xi)} - 2\mathrm{e}^{(\zeta-\frac{5}{2})\sigma(\xi)}\right)\\
&\quad + \xi E_{12}\otimes E_{21}\,\mathrm{e}^{(\zeta-\frac{1}{2})\sigma(\xi)} - \xi E_{32}\otimes E_{23}\,\mathrm{e}^{(\zeta-\frac{3}{2})\sigma(\xi)}\\
&\quad - \xi^2 E_{12}\otimes H_{13}E_{23}\,\mathrm{e}^{(\zeta-\frac{3}{2})\sigma(\xi)} + \xi^3 E_{12}^2\otimes E_{23}^2\,\mathrm{e}^{2(\zeta-\frac{3}{2})\sigma(\xi)}.
\end{aligned}\tag{28}$$

(ii) The case $\alpha = -1, \beta = 2$. There are two twists for the carrier algebra $\mathbf{L}(-1, 2)$.

First we have the coboundary form of the previous type $\omega = E^*([\,,\,])$, and the corresponding twists (23) with $\alpha = -1, \beta = 2$.

The second possibility is due to the nontrivial elements of the cohomology group $H^2\,(\mathbf{L}(-1, 2))$. The cochain

$$(\psi : \mathbf{L}\wedge\mathbf{L} \longrightarrow \mathbb{C})$$

such that

$$\psi(A, E) \neq 0$$

is not cohomologous to zero. This means that the form

$$\omega = B^*([\,,\,]) + \zeta A^*\wedge E^* \tag{29}$$

is a nontrivial cocycle for any $\zeta \in \mathbb{C}$. The corresponding twist is a composition of a Reshetikhin and deformed Jordanian [11] factors.

$$\begin{aligned}
\mathcal{F}_{\mathcal{JR}} &= \exp\left(\tfrac{1}{2}H\otimes\sigma(\zeta,\xi)\right)\exp(\zeta A\otimes E),\\
\sigma(\zeta,\xi) &= \ln\left(1 + \xi B - \tfrac{1}{2}\xi\zeta^2E^2\right).
\end{aligned}\tag{30}$$

Note that the composition is possible due to the fact that after applying the Reshetikhin factor $\mathcal{F}_{\mathcal{R}} = \exp(\zeta A \otimes E)$ we get the primitive Borel subalgebra on the space generated by $\left\{H, B - \frac{1}{2}\zeta^2 E^2\right\}$.

Here we have the universal $\mathcal{R}$-matrix:

$$\mathcal{R}_{\mathcal{JR}} = \exp\left(\tfrac{1}{2}\sigma(\zeta,\xi)\otimes H\right)\exp(\xi E\otimes A)\exp(-\xi A\otimes E)\exp\left(-\tfrac{1}{2}H\otimes\sigma(\zeta,\xi)\right). \quad (31)$$

Choosing $\zeta = \eta\xi$, we get the r-matrix

$$r_{\mathcal{JR}} = \tfrac{1}{2}H\wedge B + \eta A\wedge E. \quad (32)$$

Obviously, the term $\eta A^* \wedge E^*$ can give the nondegenerate cocycle also with the second basic coboundary $E^*([\,,\,])$. The corresponding r-matrix has the form

$$r'_{\mathcal{JR}} = H\wedge E + A\wedge B - \eta H\wedge B.$$

Nevertheless, a simple substitution ($H \to H + A$, $A \to \eta A$, $B \to -(1/\eta)(B+E)$, $E \to -E$) brings us again to the r-matrix (32). Thus, we have only two different solutions here. For the first of them, the deformed costructure can easily be obtained as a special case of (28). To present the necessary coproducts for the second case, let us use the following injection in $sl(3)$:

$$H = H_{23}, \qquad E = E_{13}, \qquad A = E_{12}, \qquad B = E_{23},$$
$$\sigma(\zeta,\xi) = \ln\left(1 + \xi E_{23} - \tfrac{1}{2}\xi\zeta^2 E_{13}^2\right).$$

In these terms, the coproducts $\Delta_{\mathcal{JR}}$ are defined by the formulae

$$\begin{aligned}
\Delta_{\mathcal{JR}}(H_{12}) &= H_{12}\otimes 1 + \tfrac{1}{2}\xi H_{23}\otimes\left(E_{23}+\zeta E_{13}^2\right)\mathrm{e}^{-\sigma(\xi,\zeta)} \\
&\quad - 3\zeta E_{12}\otimes E_{13}\,\mathrm{e}^{-\frac{1}{2}\sigma(\xi,\zeta)} + 1\otimes H_{12}, \\
\Delta_{\mathcal{JR}}(H_{23}) &= H_{23}\otimes\mathrm{e}^{-\sigma(\xi,\zeta)} + 1\otimes H_{23}, \\
\Delta_{\mathcal{JR}}(E_{12}) &= E_{12}\otimes\mathrm{e}^{-\frac{1}{2}\sigma(\xi,\zeta)} - \tfrac{1}{2}\xi H_{23}\otimes E_{13}\,\mathrm{e}^{-\sigma(\xi,\zeta)} + 1\otimes E_{12}, \\
\Delta_{\mathcal{JR}}(E_{23}) &= E_{23}\otimes\mathrm{e}^{\sigma(\xi,\zeta)} + \zeta E_{13}\otimes E_{13}\,\mathrm{e}^{\frac{1}{2}\sigma(\xi,\zeta)} + 1\otimes E_{23}, \\
\Delta_{\mathcal{JR}}(E_{13}) &= E_{13}\otimes\mathrm{e}^{\frac{1}{2}\sigma(\xi,\zeta)} + 1\otimes E_{13}, \\
\Delta_{\mathcal{JR}}(E_{21}) &= E_{21}\otimes\mathrm{e}^{\frac{1}{2}\sigma(\xi,\zeta)} - \zeta E_{12}\otimes\left(E_{23}+\zeta E_{13}^2\right)\mathrm{e}^{-\frac{1}{2}\sigma(\xi,\zeta)} \\
&\quad + \zeta H_{12}\otimes E_{13} + \tfrac{1}{2}\xi\zeta H_{23}\otimes E_{13}E_{23}\,\mathrm{e}^{-\sigma(\xi,\zeta)} + 1\otimes E_{21}, \\
\Delta_{\mathcal{JR}}(E_{32}) &= E_{32}\otimes\mathrm{e}^{-\sigma(\xi,\zeta)} + 1\otimes E_{32} \\
&\quad + \zeta E_{12}\otimes E_{12}\,\mathrm{e}^{-\frac{1}{2}\sigma(\xi,\zeta)} - \tfrac{1}{2}\xi\zeta H_{23}E_{12}\otimes E_{13}\mathrm{e}^{-\frac{3}{2}\sigma(\xi,\zeta)} \\
&\quad - \tfrac{1}{2}\xi H_{23}\otimes(H_{23} - \zeta E_{12}E_{13})\,\mathrm{e}^{-\sigma(\xi,\zeta)} \\
&\quad + \xi^2\left(\tfrac{1}{2}H_{23} - \tfrac{1}{4}H_{23}^2\right)\otimes\left(E_{23}-\zeta E_{13}^2\right)\mathrm{e}^{-2\sigma(\xi,\zeta)}, \\
\Delta_{\mathcal{JR}}(E_{31}) &= E_{31}\otimes\mathrm{e}^{-\frac{1}{2}\sigma(\xi,\zeta)} + 1\otimes E_{31} - \zeta E_{32}\otimes E_{13}\,\mathrm{e}^{-\sigma(\xi,\zeta)} \\
&\quad + \zeta E_{12}\otimes H_{13}\,\mathrm{e}^{-\frac{1}{2}\sigma(\xi,\zeta)} - \zeta^2 E_{12}^2\otimes E_{13}\,\mathrm{e}^{-\sigma(\xi,\zeta)} \\
&\quad + \tfrac{1}{2}\xi H_{23}\otimes(E_{21} + \zeta E_{13} - \zeta H_{13}E_{13})\,\mathrm{e}^{-\sigma(\xi,\zeta)} \\
&\quad + \tfrac{1}{2}\xi\zeta H_{23}E_{12}\otimes\left(-E_{23} + 2\zeta E_{13}^2\right)\mathrm{e}^{-\frac{3}{2}\sigma(\xi,\zeta)} \\
&\quad - \xi^2\zeta^2\left(\tfrac{1}{2}H_{23} - \tfrac{1}{4}H_{23}^2\right)\otimes\left(E_{23}E_{13} - \zeta E_{13}^3\right)\mathrm{e}^{-2\sigma(\xi,\zeta)}.
\end{aligned} \quad (33)$$

(iii) The case $\alpha = 0, \beta = 1$. This is the so-called peripheric case [13]. The carrier algebra $\mathbf{L}_{(0,1)}$ is defined by relations (20) with $\alpha = 0, \beta = 1$. Again we can use the same coboundary form $E^*([,])$ as in the case 1 and get the peripheric versions of the twists (23):

$$\mathcal{F}_{\mathcal{P}} = \exp(\xi A \otimes B\,\mathrm{e}^{-\sigma(\xi)}) \exp(H \otimes \sigma(\xi)),$$
$$\mathcal{F}_{\mathcal{P}'} = \exp(-\xi B \otimes A) \exp(H \otimes \sigma(\xi)) \tag{34}$$

with the costructure (for the version $\mathcal{F}_{\mathcal{P}}$)

$$\begin{aligned}
\Delta_{\mathcal{P}}(H) &= H \otimes \mathrm{e}^{-\sigma(\xi)} + 1 \otimes H - \xi A \otimes B\,\mathrm{e}^{-2\sigma(\xi)},\\
\Delta_{\mathcal{P}}(A) &= A \otimes \mathrm{e}^{\sigma(\xi)} + 1 \otimes A,\\
\Delta_{\mathcal{P}}(B) &= B \otimes \mathrm{e}^{\sigma(\xi)} + \mathrm{e}^{\sigma(\xi)} \otimes B,\\
\Delta_{\mathcal{P}}(E) &= E \otimes \mathrm{e}^{\sigma(\xi)} + 1 \otimes E.
\end{aligned} \tag{35}$$

We also have the cohomologically nontrivial map ω that can be chosen to be

$$\omega_H = H^* \wedge A^*. \tag{36}$$

The only coboundary map that can extend this ω_H to create a nondegenerate form for $\mathbf{L}(0,1)$ is again $E^*([,])$,

$$\omega = E^*([,]) + \zeta\omega_H = E^*([\,,]) + \zeta H^* \wedge A^*. \tag{37}$$

The inverse of the ω-form matrix acquires the additional term proportional to $B \wedge E$. Note that the costructure (35) provides a pair of commuting primitive elements: σ and $B\,\mathrm{e}^{-\sigma}$. This signifies the possibility of applying the corresponding Reshetikhin twist to the algebra $U_{\mathcal{P}}(\mathbf{L}(0,1))$ deformed by (34). Again for the version $\mathcal{F}_{\mathcal{P}}$ we have the composition

$$\mathcal{F}_{\mathcal{RP}} = \exp(\eta B\,\mathrm{e}^{-\sigma(\xi)} \otimes \sigma(\xi)) \exp(\xi A \otimes B\,\mathrm{e}^{-\sigma(\xi)}) \exp(H \otimes \sigma(\xi)). \tag{38}$$

Parameters ξ and η are independent. Putting $\eta = \zeta\xi$, we arrive at the r-matrix

$$r_{\mathcal{RP}} = H \wedge E + A \wedge B + \zeta B \wedge E \tag{39}$$

which is in accord with ω in (37). This construction can easily be implemented for the case $\alpha = 1, \beta = 0$ with similar results. To conclude this point, we must add that the two basic coboundary maps $E^*([,])$ and $B^*([,])$ can certainly be combined. This corresponds to the redefinition of the extension in the basic peripheric twists:

$$\mathcal{F}_{\mathcal{P}} = \exp(\xi A \otimes (B + \zeta E)\,\mathrm{e}^{-\sigma(\xi)}) \exp(H \otimes \sigma(\xi)),$$
$$\mathcal{F}_{\mathcal{P}} = \exp(-\xi(B + \zeta E) \otimes A) \exp(H \otimes \sigma(\xi)). \tag{40}$$

The latter is possible due to the equal eigenvalues of $\mathrm{ad}(H)$ on B and E.

2.6.2. Two Cartan generators. For any carrier of the type $\mathbf{L}_{\alpha,\beta}$, one can find in $\mathfrak{g}$ the element $H^{\perp}$ (that commutes with E) and as a result remains primitive after the twist $\mathcal{F}_{\mathcal{E}}$ or $\mathcal{F}_{\mathcal{P}}$. Note that in this case the extended twist multiplied by the additional Reshetikhin factor $\mathrm{e}^{\zeta H^{\perp} \otimes \sigma}$ is equivalent to the shift of the Cartan element, $H \longrightarrow H + \zeta H^{\perp}$, in the initial extended twist. (When the additional twisting element is dragged through the extension factor, the power in the exponent $\mathrm{e}^{-\beta\sigma}$ is changed because β is shifted together with H.) Thus, the additional factor does not produce new twist but results in changes of parameters of the carrier $\mathbf{L}_{\alpha,\beta}$.

In contrast, when two commuting elements are taken from $\mathbf{N}^{+}$ (for example E_{12} and E_{13}), the Cartan elements can be chosen so that the four-dimensional carrier algebra will have the structure of a direct sum of two $\mathbf{B}(2)$ subalgebras:

$$\left[H_{12}^{\perp}, E_{13}\right] = E_{13}, \qquad \left[H_{13}^{\perp}, E_{12}\right] = E_{12} \tag{41}$$

with

$$H_{12}^{\perp} = \tfrac{1}{3}(E_{11} + E_{22}) - \tfrac{2}{3}E_{33}, \qquad H_{13}^{\perp} = \tfrac{1}{3}(E_{11} + E_{33}) - \tfrac{2}{3}E_{22}.$$

Both $\mathbf{B}(2)$ subalgebras can be twisted by Jordanian twists simultaneously with independent parameters, both can be scaled (just as in the case of unique Jordanian twist).

$$\mathcal{F}_{\mathcal{J}\mathcal{J}} = \exp\left(H_{13}^{\perp} \otimes \sigma_{12}(\xi_1)\right) \exp\left(H_{12}^{\perp} \otimes \sigma_{13}(\xi_2)\right). \tag{42}$$

$$\begin{aligned}
\Delta_{\mathcal{J}\mathcal{J}}\left(H_{12}^{\perp}\right) &= H_{12}^{\perp} \otimes \mathrm{e}^{-\sigma_{13}(\xi_2)} + 1 \otimes H_{12}^{\perp}, \\
\Delta_{\mathcal{J}\mathcal{J}}\left(H_{13}^{\perp}\right) &= H_{13}^{\perp} \otimes \mathrm{e}^{-\sigma_{12}(\xi_1)} + 1 \otimes H_{13}^{\perp}, \\
\Delta_{\mathcal{J}\mathcal{J}}(E_{12}) &= E_{12} \otimes \mathrm{e}^{\sigma_{12}(\xi_1)} + 1 \otimes E_{12}, \\
\Delta_{\mathcal{J}\mathcal{J}}(E_{13}) &= E_{13} \otimes \mathrm{e}^{\sigma_{13}(\xi_2)} + 1 \otimes E_{13}, \\
\Delta_{\mathcal{J}\mathcal{J}}(E_{23}) &= E_{23} \otimes \mathrm{e}^{-\sigma_{12}(\xi_1)}\,\mathrm{e}^{\sigma_{13}(\xi_2)} + \xi H_{13}^{\perp} \otimes E_{13}\,\mathrm{e}^{-\sigma_{12}(\xi_1)} + 1 \otimes E_{23}, \\
\Delta_{\mathcal{J}\mathcal{J}}(E_{21}) &= E_{21} \otimes \mathrm{e}^{-\sigma_{12}(\xi_1)} + 1 \otimes E_{21} - \xi_2 H_{12}^{\perp} \otimes E_{23}\,\mathrm{e}^{-\sigma_{13}(\xi_2)} \\
&\quad + \xi_1 H_{13}^{\perp} \otimes H_{12}\,\mathrm{e}^{-\sigma_{12}(\xi_1)} - \xi_1 H_{12}^{\perp} H_{13}^{\perp} \otimes (\mathrm{e}^{-\sigma_{12}(\xi_1)} - \mathrm{e}^{-\sigma_{12}(\xi_1)}\,\mathrm{e}^{-\sigma_{13}(\xi_2)}) \\
&\quad + \xi_1\left(H_{13}^{\perp} - \left(H_{13}^{\perp}\right)^2\right) \otimes (\mathrm{e}^{-\sigma_{12}(\xi_1)} - \mathrm{e}^{-2\sigma_{12}(\xi_1)}), \\
\Delta_{\mathcal{J}\mathcal{J}}(E_{32}) &= E_{32} \otimes \mathrm{e}^{\sigma_{12}(\xi_1)}\,\mathrm{e}^{-\sigma_{13}(\xi_2)} + \xi_2 H_{12}^{\perp} \otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_2)} + 1 \otimes E_{32}, \\
\Delta_{\mathcal{J}\mathcal{J}}(E_{31}) &= E_{31} \otimes \mathrm{e}^{-\sigma_{13}(\xi_2)} + 1 \otimes E_{31} + \xi_2 H_{12}^{\perp} \otimes H_{13}\,\mathrm{e}^{-\sigma_{13}(\xi_2)} - \xi_1 H_{13}^{\perp} \otimes E_{32}\,\mathrm{e}^{-\sigma_{12}(\xi_1)} \\
&\quad + \xi_2\left(H_{12}^{\perp} - \left(H_{12}^{\perp}\right)^2\right) \otimes (\mathrm{e}^{-\sigma_{13}(\xi_2)} - \mathrm{e}^{-2\sigma_{13}(\xi_2)}) \\
&\quad + \xi_2 H_{12}^{\perp} H_{13}^{\perp} \otimes (\mathrm{e}^{-\sigma_{12}(\xi_1)}\mathrm{e}^{-\sigma_{13}(\xi_2)} - \mathrm{e}^{-\sigma_{13}(\xi_2)}).
\end{aligned}$$

Due to the appearance of two primitive σs (see the second two lines of the list), the general form in this case must contain additional Reshetikhin twist:

$$\mathcal{F}_{\mathcal{J}\mathcal{J}} = \exp(\zeta\sigma_{12}(\xi_1) \otimes \sigma_{13}(\xi_2)) \exp\left(H_{13}^{\perp} \otimes \sigma_{12}(\xi_1)\right) \exp\left(H_{12}^{\perp} \otimes \sigma_{13}(\xi_2)\right). \tag{43}$$

Only two of three parameters can be scaled here (to get a nontrivial contribution to the r-matrix, the parameter ζ can be chosen proportional to $1/\xi$ with $\xi_1 = \alpha_1\xi$, $\xi_2 = \alpha_2\xi$).

2.7. Six-dimensional carrier

Up to the renumeration of the basic elements, there is only one six-dimensional Frobenius subalgebra $\mathfrak{P}$ in $\mathfrak{sl}(3)$ with the generators:

$$\left\{H_{13}^{\perp}, H_{23}^{\perp}, E_{12}, E_{13}, E_{23}, E_{32}\right\}. \tag{44}$$

This subalgebra can be considered as the simplest case of parabolic subalgebras in the classical series $\mathfrak{sl}(N)$. The parabolic subalgebra arise when some negative simple root generators are dropped from the Chevalle basis of a simple Lie algebra. In our case, this happens when the generator $E_{(e_2-e_1)} = E_{21}$ is eliminated from the basis. The remaining elements generate $\mathfrak{P}$. It is easy to check that this algebra has a trivial second cohomology, $H^2(\mathfrak{P}, \mathbb{C}) = 0$. Thus, we have one solution; it is called the parabolic twist [14].

$$\begin{aligned}
\mathcal{F}_{\wp} &= \mathcal{F}_{\mathcal{D}}\mathcal{F}_{\mathcal{E}\mathcal{J}} \\
&= \exp\left(H_{13}^{\perp} \otimes (2\sigma_{13}(\xi_1) + \sigma_{32}(\xi_2)\right) \exp(-\xi_1 E_{23} \otimes E_{12}\,\mathrm{e}^{\sigma_{13}(\xi_1)}) \exp(H_{23} \otimes \sigma_{13}(\xi_1)) \qquad (45) \\
&= \exp\left(H_{13}^{\perp} \otimes \sigma_{13}(\xi_1)\right) \exp\left(H_{13}^{\perp} \otimes \sigma_{32}(\xi_2)\right) \exp\left(H_{13}^{\perp} \otimes \sigma_{13}(\xi_1)\right) \cdot \mathcal{F}_{\mathcal{E}\mathcal{J}}.
\end{aligned}$$

The parabolic twist factorizes into the ordinary extended Jordanian $\mathcal{F}_{\mathcal{EJ}}$ and the factor $\mathcal{F}_{\mathcal{D}}$. The latter can be considered as a deformed version of the Jordanian twist. The final deformed costructure in $\mathfrak{P}$ looks as follows:

$$
\begin{aligned}
\Delta_{\wp}\big(H_{13}^{\perp}\big) &= \big(H_{13}^{\perp}\otimes 1\big)(1\otimes 1+\xi_2 C(\xi_1))^{-1} + 1\otimes H_{13}^{\perp},\\
\Delta_{\wp}\big(H_{23}^{\perp}\big) &= H_{23}^{\perp}\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)} + 1\otimes H_{232}^{\perp} + \xi_1\big(E_{23}+\xi H_{12}^{\perp}H_{13}^{\perp}\big)\otimes E_{12}\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\mathrm{e}^{-\sigma_{13}(\xi_1)},\\
\Delta_{\wp}(E_{12}) &= E_{12}\otimes \mathrm{e}^{\sigma_{32}(\xi_2)}\,\mathrm{e}^{\sigma_{13}(\xi_1)} + \mathrm{e}^{\sigma_{13}(\xi_1)}\otimes E_{12} + \xi_1\xi_2 H_{13}^{\perp}E_{12}\otimes E_{12},\\
\Delta_{\wp}(E_{13}) &= \begin{pmatrix} E_{13}\otimes \mathrm{e}^{\sigma_{13}(\xi_1)}\,\mathrm{e}^{\sigma_{32}(\xi_2)} + 1\otimes E_{13}\\ -\,\xi_2 H_{13}^{\perp}\otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\end{pmatrix}(1\otimes 1+\xi_2 C(\xi_1))^{-1},\\
\Delta_{\wp}(E_{23}) &= \begin{pmatrix} E_{23}\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)} - \xi_2 H_{13}^{\perp}\otimes H_{23}\\ -\,\xi_2\big(H_{13}^{\perp}\big)^2\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)} + H_{13}^{\perp}\otimes 1\end{pmatrix}(1\otimes 1+\xi_2 C(\xi_1))^{-1}\\
&\quad -\xi_2\big(H_{13}^{\perp}(H_{13}^{\perp}-1)\otimes 1\big)(1\otimes 1+\xi_2 C(\xi_1))^{-2} + 1\otimes E_{23},\\
\Delta_{\wp}(E_{32}) &= E_{32}\otimes \mathrm{e}^{\sigma_{32}(\xi_2)} + 1\otimes E_{32} + \xi_1\big(\xi_2 E_{32} + 2\mathrm{e}^{\sigma_{32}(\xi_2)}H_{13}^{\perp}\big)\otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
&\quad +\xi_1^2\big(\xi_2^2 E_{32} + \mathrm{e}^{\sigma_{32}(\xi_2)}H_{13}^{\perp}\big)H_{13}^{\perp}\otimes (E_{12})^2(\mathrm{e}^{\sigma_{13}(\xi_1)}\,\mathrm{e}^{\sigma_{32}(\xi_2)}\,\mathrm{e}^{\sigma_{13}(\xi_1)})^{-1},
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\Delta_{\wp}(E_{21}) = \Bigg\{ & E_{21}\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)} + \xi_2 H_{13}^{\perp}\otimes E_{31} + \xi_1\xi_2\big(H_{13}^{\perp}\big)^2\otimes H_{13}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
& + \begin{pmatrix} \xi_1\xi_2\big(H_{13}^{\perp}\big)^2\\ -\,\xi_1\xi_2\big(H_{13}^{\perp}\big)^3 + \xi_1 H_{13}^{\perp}E_{23}\end{pmatrix}\otimes (\mathrm{e}^{-\sigma_{13}(\xi_1)} - \mathrm{e}^{-2\sigma_{13}(\xi_1)})\\
& + \xi_1 H_{23}^{\perp}E_{23}\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)} - \xi_1 E_{23}\otimes H_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
& + \xi_1^2\xi_2 E_{23}H_{13}^{\perp}\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}H_{23}E_{12}\\
& + \xi_1^2\xi_2 E_{23}\big(H_{13}^{\perp}\big)^2\otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-2\sigma_{13}(\xi_1)}\Bigg\}(1\otimes 1+\xi_2 C(\xi_1))^{-1}\\
& + \Big\{\xi_1^2\xi_2 E_{23}\big(H_{13}^{\perp} - \big(H_{13}^{\perp}\big)^2\big)\otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
& - \xi_1 H_{13}^{\perp}E_{23}\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)}\Big\}(1\otimes 1+\xi_2 C(\xi_1))^{-2} + \xi_1\xi_2(1\otimes 1+\xi_2 C(\xi_1))^{-1}\\
& \times\Big\{H_{12}^{\perp}\big(H_{13}^{\perp} - \big(H_{13}^{\perp}\big)^2\big)\otimes \mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\Big\} - \xi_1 H_{23}^{\perp}\otimes E_{23}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
& + \xi_1\xi_2 H_{12}^{\perp}H_{13}^{\perp}\otimes (H_{23}-1)\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
& + \big(\xi_1\xi_2 H_{12}^{\perp}\big(H_{13}^{\perp}\big)^2 - \xi_1 H_{12}^{\perp}E_{23}\big)\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
& - \xi_1^2\xi_2 H_{12}^{\perp}H_{13}^{\perp}\otimes E_{23}E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\\
& - \xi_1^2 E_{23}\otimes \mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}E_{12}E_{23}\\
& - \xi_1^2 E_{23}^2\otimes E_{12}\,\mathrm{e}^{-2\sigma_{13}(\xi_1)}(1\otimes 1+\xi_2 C(\xi_1))^{-1}(1\otimes \mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)})\\
& + 1\otimes E_{21},
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\Delta_{\wp}(E_{31}) = E_{31} \otimes \mathrm{e}^{-\sigma_{13}(\xi_1)} - \big\{&\xi_1 E_{21} + \xi_1^2\big(H_{23}^{\perp} - 1\big)E_{23} - \xi_1\xi_2 H_{13}^{\perp}E_{31} \\
&+ \xi_1^2\xi_2\big(H_{23}^{\perp} - 1\big)H_{12}^{\perp}H_{13}^{\perp}\big\} \otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)} \\
&+ \xi_1 H_{23}^{\perp} \otimes H_{13}\,\mathrm{e}^{-\sigma_{13}(\xi_1)} + \xi_1 H_{12}^{\perp}H_{13}^{\perp} \otimes (\mathrm{e}^{-\sigma_{13}(\xi_1)} - \mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}) \\
&+ \xi_1\big(H_{23}^{\perp} - \big(H_{23}^{\perp}\big)^2\big) \otimes (\mathrm{e}^{-\sigma_{13}(\xi_1)} - \mathrm{e}^{-2\sigma_{13}(\xi_1)}) \\
&+ \big(\xi_1^2 E_{23} + \xi_1^2\xi_2 H_{12}^{\perp}H_{13}^{\perp}\big) \otimes H_{13}E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)} \\
&+ \Big\{2\xi_1^2 H_{13}^{\perp}E_{23} - \xi_1^2\xi_2 H_{12}^{\perp}H_{13}^{\perp} + \xi_1^2\xi_2\big(H_{12}^{\perp}\big)^2 H_{13}^{\perp} + 2\xi_1^2\xi_2 H_{12}^{\perp}\big(H_{13}^{\perp}\big)^2\Big\} \\
&\otimes E_{12}\,\mathrm{e}^{-2\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)} + \big\{2\xi_1^2\big(H_{12}^{\perp} - 1\big)E_{23} - \xi_1^2\xi_2 H_{12}^{\perp}H_{13}^{\perp} \\
&+ \xi_1^2\xi_2\big(H_{12}^{\perp}\big)^2 H_{13}^{\perp}\Big\} \otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-2\sigma_{13}(\xi_1)} \\
&+ \Big\{2\xi_1^3\xi_2\big(H_{12}^{\perp} - 1\big)\big(H_{13}^{\perp} + 1\big)E_{23} + \xi_1^3\xi_2^2\big(H_{12}^{\perp}\big)^2\big(H_{13}^{\perp}\big)^2 \\
&- \xi_1^3\xi_2^2 H_{12}^{\perp}\big(H_{13}^{\perp}\big)^2\Big\} \otimes (E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)})^2 \\
&+ \xi_1(E_{23} \otimes \mathrm{e}^{-\sigma_{13}(\xi_1)}E_{32})(1 \otimes 1 + \xi_2 C(\xi_1))^{-1} \\
&+ \xi_1^2\big(\big(H_{13}^{\perp} + 1\big)E_{23} \otimes E_{12}\,\mathrm{e}^{-2\sigma_{13}(\xi_1)}\big)(1 \otimes 1 + \xi_2 C(\xi_1))^{-1} \\
&- \xi_1^2(1 \otimes 1 + \xi_2 C(\xi_1))^{-1}\big(H_{13}^{\perp}E_{23} \otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\,\mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}\big) \\
&+ \xi_1^3\big(E_{23}^2 \otimes E_{12}^2\,\mathrm{e}^{-2\sigma_{13}(\xi_1)}\big)(1 \otimes 1 + \xi_2 C(\xi_1))^{-1}(1 \otimes \mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}) \\
&\times (1 \otimes 1 + \xi_2 C(\xi_1)) \times (1 \otimes \mathrm{e}^{-\sigma_{32}(\xi_2)}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}) + 1 \otimes E_{31},
\end{aligned} \tag{48}
$$

where

$$
C(\xi_1) = 1 \otimes E_{32} + \xi_1 H_{13}^{\perp} \otimes E_{12}\,\mathrm{e}^{-\sigma_{13}(\xi_1)}.
$$

The parabolic twist $\mathcal{F}_{\wp}$ can be supplied with two natural parameters corresponding to two Jordanian-like factors:

$$
\mathcal{F}_{\wp}(\xi,\zeta) = \exp\big(H_{13}^{\perp} \otimes (2\sigma_{13}(\xi) + \sigma_{32}(\zeta))\big)\exp(-\xi E_{23} \otimes E_{12}\,\mathrm{e}^{\sigma_{13}(\xi)})\exp(H_{23} \otimes \sigma_{13}(\xi)). \tag{49}
$$

If in the universal $\mathcal{R}$-matrix for $U_{\wp}(\mathfrak{P};\xi,\zeta)$,

$$
\begin{aligned}
\mathcal{R}_{\wp}(\xi,\zeta) &= (\mathcal{F}_{\wp}(\xi,\zeta))_{21}(\mathcal{F}_{\wp}(\xi,\zeta))^{-1} \\
&= \exp\big((2\sigma_{13}(\xi) + \sigma_{32}(\zeta)) \otimes H_{13}^{\perp}\big)\exp(-\xi E_{12}\,\mathrm{e}^{\sigma_{13}(\xi)} \otimes E_{23})\exp(\sigma_{13}(\xi) \otimes H_{23}) \\
&\quad \times \exp(-H_{23} \otimes \sigma_{13}(\xi))\exp(\xi E_{23} \otimes E_{12}\,\mathrm{e}^{\sigma_{13}(\xi)})\exp\big(-H_{13}^{\perp} \otimes (2\sigma_{13}(\xi) + \sigma_{32}(\zeta)\big),
\end{aligned} \tag{50}
$$

the parameters are chosen to be proportional ($\zeta = \eta\xi$), expression (50) can be considered as a quantization of the classical r-matrix

$$
r_{\wp}(\eta) = H_{23}^{\perp} \wedge E_{13} + E_{12} \wedge E_{23} + \eta H_{13}^{\perp} \wedge E_{32}.
$$

3. Quantum twists for quasi-Frobenius subalgebras in $\mathfrak{sl}_3$

In the following, we define quantum deformations (q-twists) for the twists constructed in the previous section so that diagram (3) commutes. For the quantum algebra $U_q(\mathfrak{sl}_3)$, the generators will be denoted by the small letters and we shall use the following defining relations:

$$[h_{ij}, e_{kl}] = (\delta_{ik} + \delta_{jl} - \delta_{il} - \delta_{jk})e_{kl}, \qquad [e_{12}, e_{32}] = [e_{21}, e_{23}] = 0,$$
$$[e_{12}, e_{21}] = \frac{q^{h_{12}} - q^{-h_{12}}}{q - q^{-1}}, \qquad [e_{23}, e_{32}] = \frac{q^{h_{23}} - q^{-h_{23}}}{q - q^{-1}},$$
$$e_{13}e_{12} = q^{-1}e_{12}e_{13}, \qquad e_{13}e_{23} = qe_{23}e_{13},$$
$$e_{21}e_{31} = qe_{31}e_{21}, \qquad e_{32}e_{31} = q^{-1}e_{31}e_{32},$$

where the composite root generators e_{13} and e_{31} are defined as follows:

$$e_{13} := e_{12}e_{23} - q^{-1}e_{23}e_{12}, \qquad e_{31} := e_{32}e_{21} - qe_{21}e_{32}$$

and the coproduct is fixed by its values on the Chevalley generators

$$\Delta(h_{ij}) = h_{ij} \otimes 1 + 1 \otimes h_{ij},$$
$$\Delta(e_{i,i+1}) = q^{-h_{i,i+1}} \otimes e_{i,i+1} + e_{i,i+1} \otimes 1,$$
$$\Delta(e_{i+1,i}) = e_{i+1,i} \otimes q^{h_{i,i+1}} + 1 \otimes e_{i+1,i}.$$

q-twists are defined for the deformed carrier Hopf subalgebras in $U_q(\mathfrak{sl}_3)$. We consider these carrier subalgebras as q-quantization of the classical quasi-Frobenius subalgebras in $\mathfrak{sl}_3$.

3.1. Abelian two dimensional subalgebras

3.1.1. q-deformation for $\mathfrak{h}$. As far as in this case the carrier in $U_q(\mathfrak{sl}_3)$

$$H = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

is undeformed,

$$\Delta(h_{ij}) = h_{ij} \otimes 1 + 1 \otimes h_{ij},$$

the corresponding Abelian twist

$$\mathcal{F}_{\mathcal{R}} = \exp(t\xi^{ij,kl} h_{ij} \otimes h_{kl}), \qquad \xi^{ij,kl} \in \mathbb{C}, \tag{51}$$

can be taken q-independent (with the limit $\exp(t\xi^{ij,kl} H_{ij} \otimes H_{kl})$).

3.1.2. q-quantization of $\mathfrak{h}^{(1)}$. By definition,

$$\mathfrak{h}^{(1)} = * \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $U_q(\mathfrak{h}^{(1)})$ is the following Hopf subalgebra of $U_q(\mathfrak{sl}_3)$:

$$\Delta(e_{12}) = q^{-h_{12}} \otimes e_{12} + e_{12} \otimes 1, \qquad \Delta\left(h_{12}^{\perp}\right) = h_{12}^{\perp} \otimes 1 + 1 \otimes h_{12}^{\perp},$$

where $h_{12}^{\perp} = \frac{1}{3}(e_{11} + e_{22}) - \frac{2}{3}e_{33}$. The next step is to define a contraction of the algebra $U_q(\mathfrak{sl}_3)$ leading to the deformation of $U(\mathfrak{h}^{(1)})$ defined by the twisting element (see (51))

$$\mathcal{F} = \exp\left(\xi H_{\lambda}^{\perp} \otimes E_{\lambda}\right).$$

To find such a limiting procedure, we introduce a family of Hopf algebras equivalent to $U_q(\mathfrak{h}^{(1)})$. This family is obtained by applying to $U_q(\mathfrak{h}^{(1)})$ the similarity transformation defined by the coboundary twist. To fix its form, we use the notation

$$\exp_q(x) = \sum_{n\geqslant 0} \frac{x^n}{(n)_q!},$$

where

$$(n)_q = \frac{q^n - 1}{q - 1}, \qquad (n)_q! = (1)_q(2)_q \cdots (n)_q,$$

Now put $W = \exp_{q^2}(ts^{-1}e_{12})$. The necessary coboundary twist $\mathcal{J}(s, t)$ is

$$\mathcal{J}(s,t) := (W \otimes W)\Delta(W^{-1}). \tag{52}$$

According to the Heine formula [15],

$$(1 - tx)_q^{(-\alpha)} := 1 + \sum_{n\geqslant 1} t^n \frac{(\alpha)_q \cdots (\alpha + n - 1)_q}{(n)_q!} x^n$$
$$= \exp_q\left(\frac{t}{1-q}x\right)\exp_{q^{-1}}\left(-\frac{q^\alpha t}{1-q}x\right).$$

Thus, the twisting element $\mathcal{J}(s, t)$ can be simplified,

$$\mathcal{J}(s,t) = 1 \otimes 1 + \sum_{n\geqslant 1}(s^{-1}(1-q^2))^n t^n \frac{\left(-\frac{1}{2}h_{12}\right)_{q^2} \cdots \left(-\frac{1}{2}h_{12} + n - 1\right)_{q^2} \otimes e_{12}^n}{(n)_{q^2}}.$$

If we require that

$$q \equiv 1 + s^2 t \bmod(s^3 t), \tag{53}$$

then $\mathcal{J}(s, t)$ contains only positive degrees of s and moreover

$$\mathcal{J}(s,t) = 1 \otimes 1 \bmod(st). \tag{54}$$

Now consider the element

$$F_q := (W \otimes W)q^{s^{-1}\cdot h_{12}^{\perp}\otimes h_{12}}\Delta(W^{-1}) = \left(\mathrm{Ad}(W \otimes W) \circ \left(q^{s^{-1}\cdot h_{12}^{\perp}\otimes h_{12}}\right)\right)\mathcal{J}(s,t). \tag{55}$$

It defines a twist for $U_q(\mathfrak{sl}_3)$ because it satisfies the Drinfeld equation being equivalent to the Abelian twist $q^{s^{-1}\cdot h_{12}^{\perp}\otimes h_{12}}$.

Let us check that

$$F_q \equiv \exp\left(-2t^2 h_{12}^{\perp} \otimes e_{12}\right)\bmod(st). \tag{56}$$

Note that the multiplier t^2 is required as far as q-dependent terms must not contribute to the twist in the limit $s \to 0$. Using the Heine formula, we can calculate explicitly

$$\mathrm{Ad}(W \otimes W) \circ \left(q^{s^{-1}\cdot h_{12}^{\perp}\otimes h_{12}}\right) = (1 \otimes \mathrm{Ad}(W) \circ (q^{h_{12}}))^{s^{-1}\cdot h_{12}^{\perp}\otimes 1}$$

and

$$\mathrm{Ad}(W) \circ (q^{h_{12}}) = \frac{1}{1 - (1-q^2)s^{-1}t \cdot e_{12}} q^{h_{12}} \equiv (1 - 2st^2 e_{12}) \bmod(s^2 t).$$

This together with (54) and (55) proves (56). In the limit $s \longrightarrow 0$, we get the special case of the general twisting element $\mathcal{F}_{\mathcal{R}} = \exp(\xi H_\lambda^{\perp} \otimes E_\lambda)$ (see (6)).

3.1.3. q-quantization of $\mathfrak{h}^{(0,1)}$. By definition,

$$\mathfrak{h}^{(0,1)} = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To find the necessary quantized carrier $U_q(\mathfrak{h}^{(0,1)})$, let us simplify the form of the corresponding Hopf subalgebra in $U_q(\mathfrak{sl}_3)$. There, in particular, we have

$$\Delta(e_{13}) = q^{-h_{13}} \otimes e_{13} + e_{13} \otimes 1 + (1 - q^{-2}) q^{-h_{12}} e_{23} \otimes e_{12}.$$

Let us perform the twist transformation $U_q(\mathfrak{sl}_3) \longrightarrow U_{q,R_1}(\mathfrak{sl}_3)$ by applying the R-matrix factor twisting element

$$R_1 = \exp_{q^2}(-(q - q^{-1}) e_{21} \otimes e_{12}).$$

In the twisted algebra $U_{q,R_1}(\mathfrak{sl}_3)$, the coalgebra of $U_q(\mathfrak{h}^{(0,1)})$ is generated by

$$\Delta(e_{12}) = q^{h_{12}} \otimes e_{12} + e_{12} \otimes 1, \qquad \Delta(e_{13}) = q^{-h_{13}} \otimes e_{13} + e_{13} \otimes 1.$$

To define the quantum analogue of the twist $\mathcal{F}_{\mathcal{R}} = \exp(\xi E_{12} \otimes E_{13})$—the special case of the general expression (8) with $\mu = 0$—we consider the following q-twist:

$$F_q = (W \otimes W) R_1 \Delta(W^{-1}), \qquad W = \exp_{q^2}(s^{-1} t q^{-h_{12}} e_{12}) \exp_{q^2}(s^{-1} t e_{13}).$$

(Note that $[q^{-h_{12}} e_{12}, e_{13}] = 0$.) Explicitly,

$$F_q = \left(\exp_{q^2}(s^{-1} t q^{-h_{12}} e_{12}) \otimes \exp_{q^2}(s^{-1} t e_{13})\right) \exp_{q^{-2}}(-t s^{-1} q^{-h_{13}} \otimes e_{13})$$
$$\times \exp_{q^{-2}}(-s^{-1} t q^{-h_{12}} e_{12} \otimes q^{-h_{12}}) R_1.$$

Using the relation

$$\mathrm{Ad}(\exp_{q^2}(s^{-1} t q^{-h_{12}} e_{12})) \circ (q^{-h_{13}}) = (1 - (1 - q^2) s^{-1} t \cdot q^{-h_{12}} e_{12})_{q^2}^{(\frac{1}{2})} q^{-h_{13}}$$

and assuming that $q \equiv 1 + s^2 t \bmod(s^3 t)$, we can check that

$$F_q \equiv \exp(t^3 e_{12} \otimes e_{13}) \mathrm{mod}(st).$$

Another possible q-twist corresponding to the same bialgebraic structure looks like

$$F'_q = \exp_{q^2}(t\ q^{-h_{12}} e_{12} \otimes e_{13}) q^{-h_{13}^{\perp} \otimes h_{12}^{\perp}} R_1.$$

3.1.4. q-quantization of $\mathfrak{h}^{(1,1)}$ *by embedding into* $U_q\left(A_2^{(2)}\right)$. By definition,

$$\mathfrak{h}^{(1,1)} = * \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The q-quantization of the classical r-matrix $\rho = E_{13} \wedge (E_{12} + E_{23})$ can be related to the twist for $U_q(\hat{\mathfrak{sl}}_3)$. Following [16], we define here the q-twist in the root generators notation:

$$\mathcal{F}_q = \exp_{q^2}\left(\tfrac{1}{2} t\ \hat{e}_{\delta-\alpha-\beta} \otimes \hat{e}_{-\beta}\right) \exp_{q^2}(t\ \hat{e}_{\delta-\alpha-\beta} \otimes \hat{e}_{-\alpha}) \exp_{q^2}\left(-\tfrac{1}{2} q t\ \hat{e}_{\delta-\beta} \otimes \hat{e}_{-\alpha-\beta}\right)$$
$$\times \exp_{q^2}\left(-\tfrac{1}{2}(q - q^{-1})\ \hat{e}_{\alpha} \otimes \hat{e}_{-\beta}\right) \mathcal{K},$$

where

$$\mathcal{K} = q^{\frac{4}{9} h_\alpha \otimes h_\alpha + \frac{2}{9} h_\alpha \otimes h_\beta + \frac{5}{9} h_\beta \otimes h_\alpha + \frac{7}{9} h_\beta \otimes h_\beta},$$
$$\hat{e}_{-\alpha} = q^{\frac{1}{2}\ h_\beta^{\perp}} e_{-\alpha}, \qquad \hat{e}_{-\beta} = q^{-\frac{1}{2}\ h_\beta^{\perp}} e_{-\beta}, \qquad \hat{e}_{\delta-\alpha-\beta} = q^{-\frac{1}{2}\ h_{\alpha+\beta}^{\perp}} e_{\delta-\alpha-\beta},$$
$$\hat{e}_\alpha = e_\alpha q^{\frac{1}{2}\ h_{\alpha+\beta}^{\perp}}, \qquad \hat{e}'_{\delta-\beta} = [\hat{e}_\alpha, \hat{e}_{\delta-\alpha-\beta}]$$

and

$$h_\alpha^\perp = \tfrac{2}{3}h_\alpha + \tfrac{4}{3}h_\beta, \qquad h_\beta^\perp = \tfrac{4}{3}h_\alpha + \tfrac{2}{3}h_\beta, \qquad h_{\alpha+\beta}^\perp = h_\beta^\perp - h_\alpha^\perp, \qquad \hat{e}_{-\beta} = q^{-\frac{1}{2}h_\beta^\perp} e_{-\beta}.$$

In the limit $s \to 0$ and assuming that $q = 1 + st$, we come to the following twist for $U(\widehat{\mathfrak{sl}}_3)$:

$$\mathcal{F}_1 = \exp\left(\tfrac{1}{2}tE_{\delta-\alpha-\beta} \otimes E_{-\beta}\right)\exp(tE_{\delta-\alpha-\beta} \otimes E_{-\alpha})\exp\left(-\tfrac{1}{2}tE_{\delta-\beta} \otimes E_{-\alpha-\beta}\right)$$

which can be considered as an affinization of a twist quantizing the r-matrix

$$\widehat{\rho} = -\tfrac{1}{2}E_{\delta-\alpha-\beta} \wedge E_{-\beta} + \tfrac{1}{2}E_{\delta-\beta} \wedge E_{-\alpha-\beta} - E_{\delta-\alpha-\beta} \wedge E_{-\alpha}.$$

On the other hand, we can consider the quantum twisted affine Hopf algebra $U_q\left(A_2^{(2)}\right)$ (see also [17]), i.e. the Drinfeld–Jimbo quantization of the Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} = DB = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}.$$

In the evaluation representation, we have the Kac generators

$$\begin{aligned} H_0 &= -2H_{13}, & H_1 &= H_{13}, \\ E_0 &= \sqrt{2}E_{31}u, & E_1 &= (E_{12} + E_{23}), \\ F_0 &= \sqrt{2}E_{13}u^{-1}, & F_1 &= (E_{21} + E_{32}). \end{aligned}$$

The Drinfeld–Jimbo quantization $U_q\left(A_2^{(2)}\right)$ is defined by the relations

$$[h_\alpha, e_{\delta-2\alpha}] = -4e_{\delta-2\alpha}, \qquad [h_\alpha, e_\alpha] = 2e_\alpha,$$

$$[e_{\delta-2\alpha}, e_{-\delta+2\alpha}] = \frac{q^{-h_\alpha} - q^{h_\alpha}}{q - q^{-1}}, \qquad [e_\alpha, e_{-\alpha}] = \frac{q^{\frac{1}{2}h_\alpha} - q^{-\frac{1}{2}h_\alpha}}{q - q^{-1}},$$

$$\Delta(e_{\delta-2\alpha}) = q^{h_\alpha} \otimes e_{\delta-2\alpha} + e_{\delta-2\alpha} \otimes 1, \qquad \Delta(e_\alpha) = q^{-\frac{1}{2}h_\alpha} \otimes e_\alpha + e_\alpha \otimes 1,$$

$$\Delta(e_{-\delta+2\alpha}) = e_{-\delta+2\alpha} \otimes q^{-h_\alpha} + 1 \otimes e_{-\delta+2\alpha}, \ \Delta(e_{-\alpha}) = e_{-\alpha} \otimes q^{\frac{1}{2}h_\alpha} + 1 \otimes e_{-\alpha},$$

plus the Serre relations of the form

$$[[e_\alpha, e_{\delta-2\alpha}]_q, e_{\delta-2\alpha}]_q = 0, \qquad [e_\alpha, [e_\alpha, [e_\alpha, [e_\alpha, [e_\alpha, e_{\delta-2\alpha}]_q]_q]_q]_q]_q = 0,$$

where

$$[e_i, e_j]_q := e_i e_j - q^{b_{ij}} e_j e_i.$$

Let us fix the normal ordering,

$$\alpha \prec \delta + 2\alpha \prec \delta + \alpha \prec 3\delta + 2\alpha \prec 2\delta + \alpha \prec \delta \prec 2\delta - \alpha \prec 3\delta - 2\alpha \prec \delta - \alpha \prec \delta - 2\alpha,$$

and define the corresponding ordering on the set of Chevalley generators in $U\left(A_2^{(2)}\right)$:

$$E_1 \prec E_8 \prec E_5 \prec E_9 \prec E_7 \prec E_3 \prec E_4 \prec E_6 \prec E_2 \prec E_0,$$

where

$$\begin{aligned} E_2 &= \sqrt{2}(E_{21} - E_{32})u, & E_3 &= \sqrt{2}(H_{12} - H_{23})u, \\ E_4 &= -3\sqrt{2}(E_{12} - E_{23})u, & E_5 &= 6\sqrt{2}E_{13}u. \end{aligned}$$

Define the twisting element

$$F_q = (W \otimes W)\Delta(W^{-1}), \qquad W = \exp_q(ts^{-1}e_\alpha)\exp_{q^4}(ts^{-1}e_{-\delta+2\alpha}).$$

Explicitly,

$$F_q = \exp_{q^4}(ts^{-1}1 \otimes e_{-\delta+2\alpha}) \exp_{q^{-4}}(-ts^{-1}K \otimes e_{-\delta+2\alpha}) \times \exp_q(ts^{-1}1 \otimes e_\alpha) \exp_{q^{-1}}(-ts^{-1}q^{-\frac{1}{2}h} \otimes e_\alpha),$$

where

$$K := (1-(1-q)s^{-1}te_\alpha)_q^{(2)} q^{-h_\alpha}.$$

Imposing the relation $q \equiv 1 + s^2t \operatorname{mod}(s^3t)$, we can check that

$$F_q \equiv \exp(2t^3 e_\alpha \otimes e_{-\delta+2\alpha}) \operatorname{mod}(st).$$

In the limit $s \longrightarrow 0$, we come to the twisting element

$$\exp(2\sqrt{2}t^3u(E_{12}+E_{23}) \otimes E_{13})$$

that is the other special case of the general solution (8), this time with $\mu = 1$:

$$\mathcal{F}_{\mathcal{R}} = \exp(\xi(E_{12}+E_{23}) \otimes E_{13}).$$

3.2. *Non-Abelian two-dimensional subalgebras*

We have three types of nonequivalent non-Abelian quasi-Frobenius Lie subalgebras in $\mathfrak{sl}_3$

$$\mathfrak{b}^{(0)}, \quad \mathfrak{b}_\lambda, \quad \mathfrak{b}^{(1)}.$$

3.2.1. q-quantization of $\mathfrak{b}^{(0)}$. In the case $\mathfrak{b}^{(0)}$,

$$\mathfrak{b}^{(0)} = *\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + *\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{57}$$

we introduce the quantum twist

$$F_q = (W \otimes W)\Delta(W^{-1}), \qquad W = \exp_q(ts^{-1}e_\alpha).$$

Explicitly,

$$F_q = (1-(1-q)s^{-1}t1 \otimes e_\alpha)_q^{-\frac{1}{2}(h_\alpha \otimes 1)}$$

and put $q \equiv 1 + st \operatorname{mod}(s^2t)$. Then in the limit $s \to 0$ we come to the twist

$$\mathcal{F} = \exp(H_{13} \otimes \ln(1 + t(E_{12}+E_{23})))$$

(see (18)).

3.2.2. q-quantization of $\mathfrak{b}_\lambda$. By definition,

$$\mathfrak{b}_\lambda = *\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & 1-2\lambda \end{pmatrix} + \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Hopf algebra $U_q(\mathfrak{b}_\lambda)$ is the Hopf subalgebra containing e_{12} and $h_{23} - \lambda h_{12}^\perp$ with the coproducts

$$\Delta\left(h_{23} - \lambda h_{12}^\perp\right) = \left(h_{23} - \lambda h_{12}^\perp\right) \otimes 1 + 1 \otimes \left(h_{23} - \lambda h_{12}^\perp\right),$$

$$\Delta(e_{12}) = q^{2h_{23}-2\lambda h_{12}^\perp} \otimes e_{12} + e_{12} \otimes 1.$$

Note that we have the embedding $U_q(\mathfrak{b}_\lambda) \hookrightarrow U_{q,\mathcal{K}_\lambda}(\mathfrak{sl}_3)$ into the twisted algebra $U_{q,\mathcal{K}_\lambda}(\mathfrak{sl}_3)$ where the corresponding twisting element is

$$\mathcal{K}_\lambda = q^{(2\lambda-3)h_{12}^\perp \otimes h_{23}}.$$

Define the quantum twist

$$\begin{aligned} F_q &= \left(\exp_{q^2}(s^{-1}te_{12}) \otimes \exp_{q^2}(s^{-1}te_{12})\right) \Delta(\exp_{q^{-2}}(-s^{-1}te_{12})) \\ &= (1-(1-q^2)s^{-1}t1 \otimes e_{12})_{q^2}^{(-(h_{23}-\lambda h_{12}^\perp)\otimes 1)} \end{aligned}$$

and put $q \equiv 1 + st \bmod (s^2 t)$. In face of the evaluation

$$F_q \equiv \exp\left(\left(-h_{23} + \lambda h_{12}^\perp\right) \otimes \ln(1 + 2t^2 e_{12})\right) \bmod (st)\,,$$

we see that the desired quantization of the Jordanian twist (17) is obtained.

3.2.3. q-quantization of $\mathfrak{b}^{(1)}$. This case is given by the $\mathfrak{sl}_3$ subalgebra

$$\mathfrak{b}^{(1)} = * \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In $U_q(\mathfrak{sl}_3)$, we have the coproducts

$$\begin{aligned} &\Delta(e_{23}) = q^{-h_{23}} \otimes e_{23} + e_{23} \otimes 1, \\ &\Delta(e'_{13}) = q^{-h_{13}} \otimes e'_{13} + e'_{13} \otimes 1 + (q^{-1} - q) q^{-h_{23}} e_{12} \otimes e_{23}, \\ &e'_{13} = e_{12}e_{23} - q e_{23} e_{12}. \end{aligned}$$

Let us twist these coproducts by the R-matrix factor

$$R_2 = \exp_{q^2}(-(q-q^{-1})e_{32} \otimes e_{23}).$$

This leads to the simplified coalgebra

$$\Delta_{R_2}(e_{23}) = q^{h_{23}} \otimes e_{23} + e_{23} \otimes 1, \qquad \Delta_{R_2}(e'_{13}) = q^{-h_{13}} \otimes e'_{13} + e'_{13} \otimes 1.$$

Consider the twisting factor

$$\mathcal{K} = q^{-h_{13}^\perp \otimes h_{23}^\perp}, \qquad h_{13}^\perp = \tfrac{1}{3}(e_{11} + e_{33}) - \tfrac{2}{3} e_{22},$$

then we obtain the coproducts

$$\begin{aligned} &\Delta_{\mathcal{K}R_2}(q^{-h_{23}^\perp} e_{23}) = q^{-2h_{13}^\perp} \otimes q^{-h_{23}^\perp} e_{23} + q^{-h_{23}^\perp} e_{23} \otimes 1, \\ &\Delta_{\mathcal{K}R_2}(e'_{13}) = q^{-2h_{23}^\perp} \otimes e'_{13} + e'_{13} \otimes 1. \end{aligned}$$

Note that we have the relation

$$\left[q^{-h_{23}^\perp} e_{23}, e'_{13}\right] = 0.$$

$U_q(\mathfrak{b}^{(1)})$ is defined as the minimal Hopf subalgebra in $U_{q,\mathcal{K}R_2}(\mathfrak{sl}_3)$ containing $e'_{13}, e_{23}, h_{23}^\perp$.

The quantum twist with the necessary limit properties will be constructed in terms of thus defined Hopf algebra $U_q(\mathfrak{b}^{(1)})$. It contain two factors. The first one is a coboundary twist of the form

$$F_q^1 = (W \otimes W)\Delta(W^{-1}), \qquad W = \exp_{q^{-2}}\left(-s^{-1} t q^{-h_{23}^\perp} e_{23}\right) \exp_{q^2}(s^{-2} t e'_{13}).$$

Explicitly,

$$\begin{aligned} F_q^1 = &\exp_{q^{-2}}\left(-s^{-1}t1 \otimes q^{-h_{23}^\perp} e_{23}\right) \exp_{q^2}\left(s^{-1} t q^{-2h_{13}^\perp} \otimes q^{-h_{23}^\perp} e_{23}\right) \\ &\times \exp_{q^2}(s^{-2}t1 \otimes e'_{13}) \exp_{q^{-2}}\left(-s^2 t q^{-2h_{23}^\perp} \otimes e'_{13}\right). \end{aligned}$$

If

$$q \equiv 1 + s^2 t \bmod (s^3 t),$$

then

$$F_q^1 \equiv \exp\left(h_{23}^{\perp} \otimes \ln(1 + 2t^2 e'_{13})\right) \bmod (st)\,.$$

In the deformed Hopf algebra $U_{q,F_q^1}(\mathfrak{b}^{(1)})$, we have two group-like elements

$$Z_1 = W q^{2h_{23}^{\perp}} W^{-1} = q^{2h_{23}^{\perp}} \frac{1}{1 - (q^{-2} - 1)s^{-2} t e_{13}},$$
$$Z_2 = W q^{2h_{13}^{\perp}} W^{-1} = q^{2h_{13}^{\perp}} \frac{1}{1 + (q^2 - 1)s^{-1} t q^{-h_{23}^{\perp}} e_{23}}.$$

This allows us to also use the Abelian twist $q^{s^{-1}\ln(Z_1)\otimes\ln(Z_2)}$. The product

$$F_q = q^{s^{-1}\ln(Z_1)\otimes\ln(Z_2)} \cdot F_q^1$$

defines a q-twist with the property

$$F_q \equiv \exp\left(\left(h_{23}^{\perp} + e_{23}\right) \otimes \ln(1 + 2t^2 e'_{13})\right) \bmod (st). \tag{58}$$

Note that in the limit $q \longrightarrow 1$ the quantum twist structures for $\mathfrak{b}_\lambda$, $\mathfrak{b}^{(1)}$ and $\mathfrak{b}^{(0)}$ degenerate. They lead to equivalent families of ordinary Jordanian twists (13) $\mathcal{F}_{\mathcal{J}} = \exp(H \otimes \sigma(\xi)\,, \sigma(\xi) = \ln(1 + \xi E)$.

3.3. Quantum twists with four-dimensional carriers

Similar to the previous study (section 2), we consider separately the nonequivalent classes of four-dimensional Lie Frobenius subalgebras:

$$\mathfrak{r} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

and

$$\mathfrak{q}_{a_1,a_2,a_3} = * \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} + \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

We had one family of solutions associated with $\mathfrak{r}$ and three nonequivalent classes associated with a particular choice of (a_1, a_2, a_3) (see (23), (30), (34) and (43)).

3.3.1. Case $\mathfrak{r}$. Due to the isomorphism

$$\mathfrak{r} \cong \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix},$$

the case $\mathfrak{r}$ can be treated similarly to $\mathfrak{b}^{(1)}$. Define the q-twist as the coboundary twist

$$F_q = (W \otimes W)\Delta(W^{-1}),$$
$$W = \exp_{q^{-2}}\left(-s^{-1} t q^{-h_{23}^{\perp}} e_{23}\right) \exp_{q^2}(s^{-1} t e_{13})$$

and assume that $q \equiv 1 + st \bmod (s^2 t)$. Explicitly,

$$F_q = \exp_{q^{-2}}\left(-s^{-1}t1 \otimes q^{-h_{23}^{\perp}}e_{23}\right)\exp_{q^2}\left(s^{-1}tq^{-2h_{13}^{\perp}} \otimes q^{-h_{23}^{\perp}}e_{23}\right) \times \exp_{q^2}(s^{-1}t1 \otimes e'_{13})\exp_{q^{-2}}\left(-s^{-1}tq^{-2h_{23}^{\perp}} \otimes e'_{13}\right)$$

and

$$F_q \equiv \exp\left(-h_{13}^{\perp} \otimes \ln(1+2t^2e_{23})\right)\exp\left(h_{23}^{\perp} \otimes \ln(1+2t^2e_{13})\right) \bmod (st).$$

In the limit $s \longrightarrow 0$, this expression gives the double-Jordanian twist (43).

3.3.2. Case $\mathfrak{q}_{a_1,a_2,a_3}$. It was shown in table (22) that we can subdivide the case $\mathfrak{q}_{a_1,a_2,a_3}$ into the subclasses according to their cohomological properties:

$$H^2(\mathfrak{q}_{a_1,a_2,a_3}) = \begin{cases}(a_1, a_2, a_3) = (0, 1, -1), (1, 1, -2), \\ 0, \qquad \text{otherwise.}\end{cases}$$

Apply the Abelian twist

$$\mathcal{K} = q^{-(2\zeta+1)h_{13}^{\perp}\otimes h_{23}^{\perp}}$$

to the Hopf algebra $U_q(\mathfrak{sl}_3)$. Define $U_q(\mathfrak{q}_{a_1,a_2,a_3})$ as the minimal Hopf subalgebra in $U_{q,\mathcal{K}}(\mathfrak{sl}_3)$ containing e_{13}, e_{12}, e_{23} and $h_\zeta = h_{23}^{\perp} + \zeta h_{13}^{\perp}$. The coproduct of e_{13} in $U_{q,\mathcal{K}}(\mathfrak{sl}_3)$ has the form

$$\Delta(e_{13}) = q^{-2h_\zeta} \otimes e_{13} + e_{13} \otimes 1 + (1-q^2)e_{12}q^{-h_{23}} \otimes q^{-(2\zeta+1)h_{23}^{\perp}}e_{23}.$$

The q-twist is defined by the coboundary expression

$$F_q = (W \otimes W)\Delta(W^{-1}), \qquad W = \exp_{q^2}(ts^{-1}e_{13})$$

and the limit $s \to 0$ taken along the curve

$$q \equiv 1 + st \bmod (s^2 t)$$

gives two types of twists (23) and (34).

Now consider $\mathfrak{q}_{0,1,-1}$. The corresponding r-matrix has the following form:

$$r_1(\eta) = H_{23} \wedge E_{23} + 2\eta E_{12} \wedge E_{13}.$$

It is equivalent to the r-matrix

$$r_2(\lambda) = H_{23} \wedge E_{23} + \lambda(H_{23} \wedge E_{13} + E_{12} \wedge E_{23})$$

via the transformation

$$r_2(\mathrm{i}\sqrt{2\eta}) = \mathrm{Ad}\exp(\mathrm{i}\sqrt{2\eta}E_{12}) \otimes \exp(\mathrm{i}\sqrt{2\eta}E_{12}) \circ (r_1(\eta)).$$

We can propose that $U_q(\mathfrak{q}_{0,1,-1})$ is just a Hopf subalgebra in $U_q(\mathfrak{sl}_3)$ spanned by $\{e_{12}, e_{23}, q^{\pm h_{12}}, q^{\pm h_{23}}\}$. Though, it seems that there is no easy way to obtain F_q that contain the factors

$$\exp\left(\tfrac{1}{2}H_{23} \otimes \ln(1 + tE_{23} - \tfrac{1}{2}\eta^2t^3E_{13}^2)\right)\exp(\eta^2t^2E_{12} \otimes E_{13})$$

necessary to guarantee the desired properties.

3.4. Quantum twist with six-dimensional carrier

As it was mentioned above up to the conjugation the only six-dimensional subalgebra is

$$\mathfrak{p} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Let $U_q(\mathfrak{p})$ be a Hopf subalgebra in the algebra $U_{q,\mathcal{K}}(\mathfrak{sl}_3)$ obtained as a deformation of $U_q(\mathfrak{sl}_3)$ by the Abelian twist K,

$$\mathcal{K} = q^{h_{12}^{\perp}\otimes h_{23}^{\perp}}.$$

In the subalgebra $U_q(\mathfrak{p})$, we have the following coproducts:

$$\Delta_{\mathcal{K}}(e_{12}) = q^{-2h_{13}^{\perp}} \otimes e_{12} + e_{12} \otimes 1,$$
$$\Delta_{\mathcal{K}}\big(q^{-h_{23}^{\perp}}e_{23}\big) = q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}}e_{23} + q^{-h_{23}^{\perp}}e_{23} \otimes 1,$$
$$\Delta_K(e_{32}) = e_{32} \otimes q^{-2h_{13}^{\perp}} + 1 \otimes e_{32}.$$

It follows [18] that the element

$$F_{\mathcal{K}M} = \exp_{q^{-2}}((q-q^{-1})te_{32} \otimes e_{12})$$

is a twist for $U_q(\mathfrak{p})$.

Let us consider the twist F_q equivalent to $F_{\mathcal{K}M}$,

$$F_q = (W \otimes W)F_{\mathcal{K}M}\Delta(W^{-1});$$

here

$$W = \exp_{q^2}(t^2s^{-1}e_{12})\exp_{q^2}(ts^{-1}q^{-h_{23}^{\perp}}e_{23})\exp_{q^2}(t^2s^{-1}e_{12}).$$

Explicitly, we have

$$\begin{aligned} F_q = {} & \big(\exp_{q^2}\big(t^2s^{-1}e_{12}\big)\exp_{q^2}(ts^{-1}q^{-h_{23}^{\perp}}e_{23}) \otimes W\big) \\ & \times \exp_{q^{-2}}((q-q^{-1})te_{32}\otimes e_{12}) \cdot \exp_{q^{-2}}\big(-t^2s^{-1}q^{-2h_{13}^{\perp}} \otimes e_{12}\big) \\ & \times \exp_{q^{-2}}(-ts^{-1}q^{-h_{23}}e_{23} \otimes 1) \cdot \exp_{q^{-2}}\big(-ts^{-1}q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}}e_{23}\big) \\ & \times \exp_{q^{-2}}(-t^2s^{-1}e_{12}\otimes 1)\exp_{q^{-2}}\big(-t^2s^{-1}q^{-2h_{13}^{\perp}} \otimes e_{12}\big). \end{aligned}$$

To transform F_q further we use the commutation property

$$\big[q^{-h_{23}^{\perp}}e_{23}, e_{32}\big] = \frac{q^{-2h_{13}^{\perp}} - q^{-2h_{12}^{\perp}}}{q-q^{-1}}$$

and the relations

$$\begin{aligned} & \exp_{q^{-2}}((q-q^{-1})te_{32}\otimes e_{12})\exp_{q^{-2}}\big(-ts^{-1}q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}}e_{23}\big) \\ & \quad = \exp_{q^{-2}}\big(-ts^{-1}q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}}e_{23}\big)\exp_{q^{-2}}\big((1-q^2)s^{-1}t^2e_{32}q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}}e'_{13}\big) \\ & \qquad \times \exp_{q^{-2}}((q-q^{-1})te_{32}\otimes e_{12}). \end{aligned}$$

As a result, the corresponding factors in F_q can be transposed,

$$\begin{aligned} & \exp_{q^{-2}}((q-q^{-1})te_{32}\otimes e_{12})\exp_{q^{-2}}\big(-t^2s^{-1}q^{-2h_{13}^{\perp}} \otimes e_{12}\big)\exp_{q^{-2}}\big(-ts^{-1}q^{-h_{23}^{\perp}}e_{23}\otimes 1\big) \\ & \quad = \exp_{q^{-2}}(-ts^{-1}q^{-h_{23}}e_{23}\otimes 1)\exp_{q^{-2}}\big(-t^2s^{-1}q^{-2h_{12}^{\perp}} \otimes e_{12}\big) \\ & \qquad \times \exp_{q^{-2}}((q-q^{-1})te_{32}\otimes e_{12}), \end{aligned}$$

and the twisting element takes the form

$$F_q = (1 \otimes W) \exp_{q^{-2}}\left(-t^2 s^{-1} q^{-2h_{12}^{\perp}} \otimes e_{12}\right) \exp_{q^{-2}}\left(-ts^{-1} q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}} e_{23}\right)$$
$$\times \exp_{q^{-2}}\left((1-q^2)s^{-1}t^2 e_{32} q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}} e'_{13}\right)$$
$$\times \exp_{q^{-2}}((q-q^{-1})te_{32} \otimes e_{12}) \exp_{q^{-2}}\left(-t^2 s^{-1} q^{-2h_{13}^{\perp}} \otimes e_{12}\right).$$

Now taking into account that the following commutator is zero,

$$\left[\exp_{q^2}(ts^{-1} 1 \otimes q^{-h_{23}} e_{23}) \exp_{q^2}(t^2 s^{-1} 1 \otimes e_{12}),\right.$$
$$\left.\exp_{q^{-2}}\left(-t^2 s^{-1} q^{-2h_{12}^{\perp}} \otimes e_{12}\right) \exp_{q^{-2}}\left(-ts^{-1} q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}} e_{23}\right)\right] = 0,$$

the final expression for the q-twisting element can be obtained:

$$F_q = \exp_{q^2}(t^2 s^{-1} 1 \otimes e_{12}) \exp_{q^{-2}}\left(-t^2 s^{-1} q^{-2h_{12}^{\perp}} \otimes e_{12}\right)$$
$$\times \exp_{q^2}\left(ts^{-1} 1 \otimes q^{-h_{23}^{\perp}} e_{23}\right) \cdot \exp_{q^{-2}}\left(-ts^{-1} q^{-2h_{12}^{\perp}} \otimes q^{-h_{23}^{\perp}} e_{23}\right)$$
$$\times \exp_{q^{-2}}\left((1-q^2)s^{-1}t^2 e_{32} q^{-2h_{23}^{\perp}} \otimes q^{-h_{23}^{\perp}} e'_{13}\right)$$
$$\times \exp_{q^{-2}}((q-q^{-1})te_{32} \otimes e_{12})$$
$$\times \exp_{q^2}(t^2 s^{-1} 1 \otimes e_{12}) \exp_{q^{-2}}\left(-t^2 s^{-1} q^{-2h_{13}^{\perp}} \otimes e_{12}\right).$$

Assuming $q \equiv 1 + st \bmod(s^2 t)$ and applying the Heine formula, we can calculate the limit $s \to 0$ which gives the twist

$$\exp\left(H_{12}^{\perp} \otimes \ln(1 + 2t^3 E_{12})\right) \exp\left(H_{12}^{\perp} \otimes \ln(1 + 2t^2 E_{23})\right)$$
$$\times \exp\left(-2t^3 E_{32} \otimes E_{13}\right) \exp\left(H_{13}^{\perp} \otimes \ln(1 + 2t^3 E_{12})\right).$$

This expression is the special case of the parabolic twist obtained in [14] and presented here in section 2 (see (49)).

4. Conclusions

We have shown that the factorization property presents the possibility of obtaining all the solutions to the twist equations for algebra $U(sl(3))$. The full list of the antisymmetric classical r-matrices, constant solutions of CYBE, was quantized and the corresponding twists were constructed explicitly in the form of product of twisting factors. Each of these factors refers to an independent solution of the twist equation.

We have also demonstrated that when the Drinfeld–Jimbo r-matrix and the antisymmetric r-matrix corresponding to the twist are compatible (that is their sum gives rise to a solution for the modified classical Yang–Baxter equation) the quantum counterpart of this twist can be obtained.

It is known that triangular twists permit us to deform integrable models related to Yangians [19]. We suppose that constructed coboundary q-analogues of triangular twists give rise to a possibility of studying above-mentioned deformed models starting directly from known anisotropic models. The latter being connected with the corresponding quantum affine algebras are similarly transformed under coboundary twists. Hence, a new basis of eigenvectors will appear.

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Appendix

Contrary to the situation described in the beginning of the subsection 2.6.1, the peripheric carriers $\mathbf{L}_{1,0}$ ($\mathbf{L}_{0,1}$) present more interesting possibilities—the twisting elements can be enlarged by the additional factors. These constructions can be proved to be equivalent to the ordinary double-Cartan case described in 1.3.2 but deserve separate presentation. Here the carrier $\mathbf{L}_{1,0}$ is more convenient for our purposes and the peripheric twist looks like

$$\mathcal{F}_{\mathcal{P}} = \exp(\xi E_{12} \otimes E_{23}) \exp(H_{\mathcal{P}} \otimes \sigma(\xi)), \tag{A.1}$$

with

$$H_{\mathcal{P}} = \tfrac{2}{3}E_{11} - \tfrac{1}{3}E_{22} - \tfrac{1}{3}E_{33}, \qquad \sigma(\xi) = \ln(1 + \xi E_{13}). \tag{A.2}$$

The costructure can be obtained from (24):

$$\begin{aligned} \Delta_{\mathcal{P}}(H_{\mathcal{P}}) &= H_{\mathcal{P}} \otimes \mathrm{e}^{-\sigma} + 1 \otimes H_{\mathcal{P}} - E_{12} \otimes E_{23}\, \mathrm{e}^{-\sigma}, \\ \Delta_{\mathcal{P}}(E_{12}) &= E_{12} \otimes 1 + 1 \otimes E_{12}, \\ \Delta_{\mathcal{P}}(E_{23}) &= E_{23} \otimes + \mathrm{e}^{\sigma} \otimes E_{23}, \\ \Delta_{\mathcal{P}}(E_{13}) &= E_{13} \otimes \mathrm{e}^{\sigma} + 1 \otimes E_{13}. \end{aligned} \tag{A.3}$$

The element E_{12} remains primitive. Now in $\mathfrak{g}$ there is a Cartan element

$$H^{\perp} = \tfrac{1}{3}E_{11} - \tfrac{2}{3}E_{22} + \tfrac{1}{3}E_{33}, \tag{A.4}$$

whose dual is orthogonal to $\lambda = e_1 - e_3$. Consequently, this element also remains primitive after the peripheric twist $\mathcal{F}_{\mathcal{P}}$. The corresponding Borel subalgebra $\mathbf{B}(2)$ with the generators $H^{\perp}, E_{12}$ can be twisted additionally by the Jordanian twist (13). As a result, the triple of twisting factors form a twist

$$\mathcal{F}_{\mathcal{J}\mathcal{P}\mathcal{J}} = \exp(H^{\perp} \otimes \sigma_{12}(\xi)) \exp(\xi E_{12} \otimes E_{23}) \exp(H_{\mathcal{P}} \otimes \sigma(\xi)), \tag{A.5}$$

with

$$\sigma_{12}(\xi) = \ln(1 + \xi E_{12}). \tag{A.6}$$

Still, the carrier algebra for this twist cannot have the dimension greater than 4. In the classical r-matrix, the additional term originating from the factor $\exp(\xi H^{\perp} \otimes \sigma_{12})$ induces a change of the E_{23} basic element for $B = E_{23} - \xi H^{\perp}$. With this change, the four-dimensional space of the carrier generated by $\left\{H_{\mathcal{P}}, E_{12}, E_{13}, B = H^{\perp} - \frac{1}{\xi}E_{23}\right\}$ becomes closed under the compositions of $\mathfrak{g}$. Thus, we obtain the deformation $\mathbf{L}_{1,0}^{\mathrm{def}}$ with the relations

$$\begin{aligned} &[H_{\mathcal{P}}, E_{12}] = E_{12}, \qquad [H_{\mathcal{P}}, B] = 0, \\ &[H_{\mathcal{P}}, E_{13}] = E_{13}, \qquad [E_{12}, E_{13}] = 0, \\ &[E_{12}, B] = E_{13} + \xi E_{12}. \end{aligned} \tag{A.7}$$

Despite the fact that the deforming function μ with $\mu(E_{12}, B) = E_{12}$ is a coboundary ($\mu \in B^2(\mathbf{L}_{1,0}, \mathbf{L}_{1,0})$), this deformation is nontrivial. This can be checked by inspecting the ranks: $\operatorname{rank}(\mathbf{L}_{1,0}) = 1$ and $\operatorname{rank}\left(\mathbf{L}_{1,0}^{\mathrm{def}}\right) = 2$. (Note that the similarity transformation that cancels μ brings the cohomologically nontrivial term in the second order of the deformation parameter.) Thus, incorporating the generator $H^{\perp}$ in the structure of the twist we have passed to the new carrier $\mathbf{L}_{1,0}^{\mathrm{def}}$. The latter must be identified with a Frobenius subalgebra in $\mathfrak{g}$. To find such consider the new basis $\{H = H_{\mathcal{P}} - B, B, A = \xi E_{12} + E_{13}, E = E_{13}\}$. Now the commutation relations are

$$\begin{aligned} &[H, A] = 0, \qquad [B, A] = A, \qquad [H, B] = 0, \\ &[H, E] = E, \qquad [B, E] = 0, \qquad [A, E] = 0. \end{aligned} \tag{A.8}$$

Thus, $\mathbf{L}_{1,0}^{\text{def}} = \mathbf{B}(2) \oplus \mathbf{B}(2)$ (the structure that we had in (41)). Obviously, having this form for $\mathbf{L}_{1,0}^{\text{def}}$ we can apply to it the double-Jordanian twist (42)

$$\mathcal{F}_{\mathcal{J}\mathcal{J}} = \exp(H \otimes \sigma_E) \exp(B \otimes \sigma_A), \qquad \sigma_A = \ln(1 + A), \sigma_E = \ln(1 + E). \tag{A.9}$$

Returning to the initial basis in $\mathfrak{g}$, it can be written as

$$\begin{aligned}\mathcal{F}_{\mathcal{J}\mathcal{J}} &= \exp\big(H^{\perp} \otimes \big(\sigma_{(\xi E_{12}+E_{13})} - \sigma_{13}\big) - \frac{1}{\xi} E_{23} \otimes \big(\sigma_{(\xi E_{12}+E_{13})} - \sigma_{13}\big)\big) \exp(H_{\mathcal{P}} \otimes \sigma_{13}) \\ &= \exp\big(H^{\perp} \otimes \big(\sigma_{(\xi E_{12}+E_{13})} - \sigma_{13}\big)\big) \exp(-E_{23} \otimes E_{12}\, \mathrm{e}^{-\sigma_{13}}) \exp(H_{\mathcal{P}} \otimes \sigma_{13}) \\ &= \exp(H^{\perp} \otimes \sigma_{(\xi E_{12}+E_{13})}) \exp(-E_{23} \otimes E_{12}) \exp((H_{\mathcal{P}} - H^{\perp}) \otimes \sigma_{13}).\end{aligned} \tag{A.10}$$

The form of the twisting element looks similar to that of (A.5) but is different. Note that here the first two twisting factors present the peripheric twist that produces the primitive coproduct for the element E_{23}. So the last quasi-Jordanian factor is based on the quasiprimitive combination of elements. The r-matrix is the same as in the case (A.5).

References

[1] Reshetihkin N Yu, Takhtajan L A and Faddeev L D 1989 *Algebra i analiz* **1** 178–206

[2] Chaichian M, Kulish P P, Nishijima K and Tureanu A 2004 *Phys. Lett.* B **273** 98–104 (*Preprint* hep-th/0408069)

[3] Aschieri P, Dimitrijevic M, Meyer F and Wess J 2005 Noncommutative geometry and gravity *Preprint* hep-th/0510059

[4] Drinfeld V G 1983 *Dokl. Acad. Nauk* **273** 531–5

[5] Bonneau P, Gerstenhaber M, Giaquinto A and Sternheimer D 2004 *J. Math. Phys.* **45** 3703–41

[6] Stolin A 1991 *Math. Scand.* **69** 57–80
Stolin A 1991 *Math. Scand.* **69** 81–8

[7] Reshetikhin N Yu 1990 *Lett. Math. Phys.* **20** 331–5

[8] Ogievetsky O V 1994 *Rend. Cir. Math. Palermo 2) Suppl.* **37** 185–99

[9] Giaquinto A and Zhang J J 1998 *J. Pure Appl. Algebra* **128** 133 (*Preprint* hep-th/9411140)

[10] Kulish P P, Lyakhovsky V D and Mudrov A I 1999 *J. Math. Phys.* **40** 4569–86

[11] Kulish P P and Lyakhovsky V D 2000 *J. Phys. A Math. Gen.* **33** L279–85

[12] Lyakhovsky V D 2004 *Supersymmetries and Quantum Symmetries* (Singapore: World Scientific) pp 120–30

[13] Lyakhovsky V D and del Olmo M A 1999 *J. Phys. A: Math. Gen.* **32** 4541–52

[14] Lyakhovsky V D and Samsonov M E 2002 *J. Algebra Appl.* **1** 413–24

[15] Kac V and Cheung P 2002 *Quantum Calculus* (Berlin: Springer)

[16] Samsonov M 2005 *Lett. Math. Phys.* **72** 197–210

[17] Lukierski J and Tolstoy V N 1997 *Czech. J. Phys.* **47** 1231–40

[18] Kulish P P and Mudrov A I 1999 *Lett. Math. Phys.* **47** 139–48 (*Preprint* math.QA/9804006)

[19] Kulish P P and Stolin A A 1997 *Czech. J. Phys.* **47** 1207–12